SURJECTIVITY OF THE TOTAL CLIFFORD INVARIANT AND BRAUER DIMENSION

ASHER AUEL

Abstract. A celebrated theorem of Merkurjev—that the 2-torsion of the Brauer group is represented by Clifford algebras of quadratic forms—is in general false when the base is no longer a field. The first counterexamples, when the base is among certain arithmetically subtle hyperelliptic curves over local fields, were constructed by Parimala, Scharlau, and Sridharan. We prove that considering Clifford algebras of all line bundle-valued quadratic forms, such counterexamples disappear and we recover Merkurjev's theorem in these cases: for any smooth curve over a local field or any smooth surface over a finite field, the 2-torsion of the Brauer group is always represented by Clifford algebras of line bundle-valued quadratic forms.

Introduction

A consequence of Merkurjev's celebrated result [Mer1]—settling the degree 2 case of the Milnor conjecture—is that every 2-torsion Brauer class over a field of characteristic $\neq 2$ is represented by the Clifford algebra of a quadratic form. There are many alternate proofs of Merkurjev's theorem [Ara], [Mer2], [Wad], [EKM, VIII], and it retains its status as one of the great breakthroughs in the theory of quadratic forms in the second half of the 20th century.

There have been many investigations into the validity of aspects of the Milnor conjecture over more general rings. For example, see [Gui, §3], [EVMS], [Hoo], and [KMS], [Ker] for a Milnor K-theoretic perspective, [PS1], [Mor], and [Gil] for a Witt group perspective, and [Auel3] for a survey of results. In this context, Alex Hahn asked if there exists a commutative ring R over which the analogue of Merkurjev's theorem doesn't hold, i.e., ${}_2\mathrm{Br}(R)$ is not represented by Clifford algebras of regular quadratic forms over R. The surprising results of Parimala, Scharlau, and Sridharan [PSch], [PS1], [PS1] show that for a smooth complete hyperelliptic curve X with a rational point over a local field of characteristic $\neq 2$, the analogue of Merkurjev's theorem over X holds if and only if X has a rational theta characteristic (which can fail to happen). These examples are also used to construct affine schemes over which Merkurjev's theorem does not hold, thus answering Hahn's original question.

In this work we show that even when Brauer classes of period 2 over a given scheme X cannot be represented by Clifford algebras of regular quadratic forms over X, they may be represented by Clifford algebras of regular line bundle-valued quadratic forms. Let $W_{\mathrm{tot}}(X)$ be the total Witt group of line bundle-valued quadratic forms (see §2.1 for definitions) and let $I^2_{\mathrm{tot}}(X)$ be the subgroup of line bundle-valued quadratic forms of even rank and trivial discriminant. We construct (in §2.4) a natural group homomorphism $e^2 : I^2_{\mathrm{tot}}(X) \to {}_2\mathrm{Br}(X)$ with values in the 2-torsion of the Brauer group of X, generalizing the classical Clifford invariant, and which we call the *total Clifford invariant*. A succinct consequence of our main result is the following.

Theorem A. *Let X be a smooth curve over a local field of characteristic $\neq 2$ or a smooth surface over a finite field of odd characteristic. Then the total Clifford invariant*

$$e^2 : I^2_{\mathrm{tot}}(X) \to {}_2\mathrm{Br}(X)$$

is surjective. In other words, every 2-torsion Brauer class on X is represented by the Clifford algebra of a regular line bundle-valued quadratic form on X.

2010 *Mathematics Subject Classification.* Primary 11E81, 11E88, 14F22; Secondary 19G12, 20G35.

Key words and phrases. Quadratic forms, line bundle-valued quadratic forms, Clifford algebras, Brauer groups, Brauer dimension, classical invariants, cohomological invariants.

In the proof (see §3), we apply results of Saltman [Sal2] and Lieblich [Lie2] on the Brauer dimension of function fields of curves over local fields and surfaces over finite fields, respectively. Together with a purity result for division algebras on surfaces (Theorem 3.6), we reduce the problem to one concerning Azumaya algebras of degree dividing 4 and index dividing 2. Then we generalize results of Knus, Ojanguren, Parimala, Paques, and Sridharan [KOS], [KP], [Knu1], [KPS1], [KPS2], and [BK] (also see [KMRT, IV §15]), on the exceptional isomorphisms of Dynkin diagrams $\mathsf{A}_1^2 = \mathsf{D}_2$ and $\mathsf{A}_3 = \mathsf{D}_3$, which provide beautiful constructions of line bundle-valued quadratic forms with specified even Clifford algebras. In fact, our main result (Theorem 3.5) applies to any regular integral scheme X satisfying purity and Brauer dimension bounded by 2 for algebras of period 2 over the function field.

The verification that the total Clifford invariant is well defined is no small task, and occupies the bulk of §1–2. The majority of the work goes into establishing two fundamental algebraic structural results: the Brauer triviality of the even Clifford algebra of a line bundle-valued metabolic form (Theorem 1.7), generalizing the main result of [KO]; and a formula to compute the even Clifford algebras and bimodules of orthogonal sums (Theorem 1.8) leading to a generalization of the classical fundamental relation in the Brauer group (Theorem 2.6). To this end, we use a new direct tensorial construction of the even Clifford algebra and bimodule (see §1.2), which offers novel universal properties (Propositions 1.1 and 1.4) useful in establishing these results. These structural results for line bundle-valued forms are new and are useful in a variety of contexts. In particular, they go beyond the author's previous cohomological construction [Auel1] of Clifford-type invariants.

History. The notion of a line bundle-valued quadratic form on X appeared in many different contexts in the early 1970s. Geyer–Harder–Knebusch–Scharlau [GHKS] introduced the notion of symmetric bilinear forms with values in the module of Kähler differentials over a global function field. This notion enables a consistent choice of local traces in order to generalize residue theorems to nonrational function fields. For a smooth complete algebraic curve X, Mumford [Mum] introduced the notion of locally free $\mathscr{O}_X$-modules with pairings taking values in the sheaf of differentials ω_X to study theta characteristics. Kanzaki [Kan] introduced the notion of a Witt group of quadratic forms with values in an invertible module over a commutative ring. Saltman [Sal1, Thm. 4.2] showed that involutions on Azumaya algebras naturally lead to the consideration of line bundle-valued bilinear forms.

In his thesis, Bichsel [Bic] (reported in [BK]) constructed an even Clifford algebra of a line bundle-valued quadratic form. This was later used in [PS2], [Bal], and [Voi]. Kapranov [Kap, §4.1] constructed a *homogeneous* Clifford algebra of a quadratic form—which in hindsight is related to the *generalized* Clifford algebra of [BK] or the *graded* Clifford algebra of [CvO]—to study the derived category of projective quadrics and quadric fibrations. This was further developed by Kuznetsov [Kuz]. With respect to Clifford algebras, line bundle-valued quadratic forms behave much like Azumaya algebras with orthogonal involutions, which do not enjoy a "full" Clifford algebra, only an even part together with a bimodule. In particular, line bundle-valued quadratic forms have no Clifford invariant in the classical sense. The construction of secondary invariants in étale cohomology capturing the even Clifford algebra of a line bundle-valued quadratic form with fixed discriminant appeared in [Auel1]. In the present work, we develop a purely algebraic Clifford invariant for line bundle-valued quadratic forms with trivial discriminant, taking values in the 2-torsion of the Brauer group.

Acknowledgments. The author would like to thank the directors of the Max-Planck-Institut für Mathematik in Bonn for providing excellent working conditions and a wonderfully stimulating environment where much of this work was accomplished. A visit at the Forschungsinstitut für Mathematik at ETH Zürich also proved to be very fruitful. The author would also personally like to thank B. Calmès, T. Chinburg, M. Knus, R. Parimala, D. Saltman, and V. Suresh for many useful conversations and much encouragement. Author partially supported by National Science Foundation MSPRF grant DMS-0903039 and an NSA Young Investigator Grant.

1. Line bundle-valued quadratic forms and even Clifford algebras

Let X be a separated noetherian scheme. By a *vector bundle*, we mean a locally free $\mathscr{O}_X$-module of constant finite rank. Fix a *line bundle* $\mathscr{L}$ on X, i.e., an invertible $\mathscr{O}_X$-module.

1.1. **Line bundle-valued quadratic forms.** A *(line bundle-valued) symmetric bilinear form* on X is a triple $(\mathscr{E}, b, \mathscr{L})$, where $\mathscr{E}$ is a vector bundle on X and $b : S^2\mathscr{E} \to \mathscr{L}$ is an $\mathscr{O}_X$-module morphism. A *(line bundle-valued) quadratic form* on X is a triple $(\mathscr{E}, q, \mathscr{L})$, where $\mathscr{E}$ is a vector bundle on X and $q : \mathscr{E} \to \mathscr{L}$ is an $\mathscr{O}_X$-homogeneous morphism of degree two such that the associated morphism $b_q : S^2\mathscr{E} \to \mathscr{L}$ defined on sections by $b_q(vw) = q(v+w) - q(v) - q(w)$ is a symmetric bilinear form. We will mostly dispense with the title "line bundle-valued." The *rank* of $(\mathscr{E}, q, \mathscr{L})$ is the rank of $\mathscr{E}$.

A symmetric bilinear form $(\mathscr{E}, b, \mathscr{L})$ is *regular* if the canonical adjoint $\psi_b : \mathscr{E} \to \mathscr{H}om(\mathscr{E}, \mathscr{L})$ is an isomorphism. A quadratic form q is *regular* if b_q is regular. If 2 is assumed invertible on X, then we can pass back and forth between quadratic and symmetric bilinear forms on X.

A *similarity transformation* between symmetric bilinear forms $(\mathscr{E}, b, \mathscr{L})$ and $(\mathscr{E}', b', \mathscr{L}')$ or quadratic forms $(\mathscr{E}, q, \mathscr{L})$ and $(\mathscr{E}', q', \mathscr{L}')$ is a pair (φ, λ) consisting of $\mathscr{O}_X$-module isomorphisms $\varphi : \mathscr{E} \to \mathscr{E}'$ and $\lambda : \mathscr{L} \to \mathscr{L}'$ such that $b'(\varphi(v), \varphi(w)) = \lambda \circ b(v, w)$ or $q'(\varphi(v)) = \lambda \circ q(v)$ on sections, respectively. A similarity transformation (φ, λ) is an *isometry* if $\mathscr{L} = \mathscr{L}'$ and λ is the identity map.

Denote by $\mathbf{GO}(\mathscr{E}, q, \mathscr{L})$ (resp. $\mathbf{O}(\mathscr{E}, q, \mathscr{L})$) the presheaf, on the large fppf site X_{fppf}, of similitudes (resp. isometries) of a regular quadratic form $(\mathscr{E}, q, \mathscr{L})$. In fact, this is a sheaf and is representable by a smooth affine reductive group scheme over X; see [DG, II.1.2.6, III.5.2.3]). Here we consider reductive group schemes whose fibers are not necessarily geometrically integral, in contrast to [SGA3, XIX.2]. In particular, the pointed nonabelian cohomology set $H^1_{\mathrm{fppf}}(X, \mathbf{GO}(\mathscr{E}, q, \mathscr{L}))$ is in bijection with the similarity classes of regular line bundle-valued quadratic forms with the same rank as $(\mathscr{E}, q, \mathscr{L})$; see [Auel1, Prop. 1.2]. If n is even or 2 is invertible on X, then the fppf site can be replaced by the étale site.

Define the *projective similarity* class of a quadratic form $(\mathscr{E}, q, \mathscr{L})$ to be the set of similarity classes of quadratic forms $(\mathscr{N} \otimes \mathscr{E}, q_{\mathscr{N}} \otimes q, \mathscr{N}^{\otimes 2} \otimes \mathscr{L})$ ranging over all regular bilinear forms $(\mathscr{N}, q_{\mathscr{N}}, \mathscr{N}^{\otimes 2})$ of rank 1 on X. In [BC], this is referred to as a *lax-similarity* class. In their notation, a *quadratic alignment* $A = (\mathscr{N}, \phi)$ between line bundles $\mathscr{L}$ and $\mathscr{L}'$ consists of a line bundle $\mathscr{N}$ and an $\mathscr{O}_X$-module isomorphism $\phi : \mathscr{N}^{\otimes 2} \otimes \mathscr{L} \to \mathscr{L}'$. A quadratic alignment induces an equivalence $A^{\circlearrowleft}$ between categories of $\mathscr{L}$-valued and $\mathscr{L}'$-valued quadratic forms (in particular, an isomorphism $A^{\circlearrowleft} : W(X, \mathscr{L}) \to W(X, \mathscr{L}')$ of Witt groups) defined by $A^{\circlearrowleft} : (\mathscr{E}, q, \mathscr{L}) \mapsto (\mathscr{N} \otimes \mathscr{E}, \phi \circ (q_{\mathscr{N}} \otimes q), \mathscr{L}')$, where $q_{\mathscr{N}} : \mathscr{N} \to \mathscr{N}^{\otimes 2}$ is the canonical squaring form.

1.2. **Even Clifford algebra.** In his thesis, Bichsel [Bic] constructs an even Clifford algebra of a line bundle-valued quadratic form on an affine scheme. Alternate constructions are given in [BK], [CvO], and [PS2, §4], which are all detailed in [Auel1, §1.8]. Inspired by [KMRT, II Lemma 8.1, §9], we now give a direct tensorial construction. Let $(\mathscr{E}, q, \mathscr{L})$ be a line bundle-valued quadratic form on X

Define ideals $\mathscr{J}_1$ and $\mathscr{J}_2$ of the tensor algebra $T(\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee)$ to be locally generated by

$$v \otimes v \otimes f - f(q(v)) \cdot 1 \quad \text{and} \quad u \otimes v \otimes f \otimes v \otimes w \otimes g - f(q(v))\, u \otimes w \otimes g, \tag{1}$$

respectively, for sections u, v, w of $\mathscr{E}$ and f, g of $\mathscr{L}^\vee$. We define

$$\mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) = T(\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee)/(\mathscr{J}_1 + \mathscr{J}_2) \tag{2}$$

together with the canonically induced morphism of $\mathscr{O}_X$-modules

$$i : \mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee \to \mathscr{C}_0(\mathscr{E}, q, \mathscr{L}), \tag{3}$$

which factors through the degree one elements of the tensor algebra.

Writing the rank as $n = 2m$ or $n = 2m+1$, there is a filtration by $\mathscr{O}_X$-modules

$$\mathscr{O}_X = \mathscr{F}_0 \subset \mathscr{F}_2 \subset \cdots \subset \mathscr{F}_{2m} = \mathscr{C}_0(\mathscr{E}, q, \mathscr{L}),$$

where $\mathscr{F}_{2i}$ is the image of the truncated tensor algebra $T^{\leq i}(\mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee)$ in $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$, for each $0 \leq i \leq m$. As in [Knu2, IV §1.6], this filtration has associated graded pieces $\mathscr{F}_{2i}/\mathscr{F}_{2(i-1)} \cong \bigwedge^{2i} \mathscr{E} \otimes (\mathscr{L}^\vee)^{\otimes i}$. In particular, $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ is a locally free $\mathscr{O}_X$-algebra of rank 2^{n-1}. By its tensorial construction, the even Clifford algebra has the following.

Proposition 1.1 (Universal Property of the even Clifford algebra)**.** *Given an $\mathscr{O}_X$-algebra $\mathscr{A}$ and an $\mathscr{O}_X$-module morphism $j : \mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee \to \mathscr{A}$ such that*

$$j(v \otimes v \otimes f) = f(q(v)) \cdot 1 \quad \textit{and} \quad j(u \otimes v \otimes f) \cdot j(v \otimes w \otimes g) = f(q(v))\, j(u \otimes w \otimes g),$$

then there exists a unique $\mathscr{O}_X$-algebra homomorphism $\psi : \mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \to \mathscr{A}$ satisfying $j = \psi \circ i$.

A similar universal property for algebras with involution over a field is stated in [Mah, §3]. The even Clifford algebra has the following additional properties.

Proposition 1.2. *Let $(\mathscr{E}, q, \mathscr{L})$ be a regular quadratic form of rank n on a scheme X. Write $n = 2m$ or $n = 2m+1$.*

a) If n is odd, $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ is a central $\mathscr{O}_X$-algebra. If n is even, the center $\mathscr{Z}(\mathscr{E}, q, \mathscr{L})$ of $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ is an étale quadratic $\mathscr{O}_X$-algebra.

b) If n is odd, $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ is an Azumaya $\mathscr{O}_X$-algebra of degree 2^m. If n is even, $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ is an Azumaya $\mathscr{Z}(\mathscr{E}, q, \mathscr{L})$-algebra of rank 2^{m-1}

c) The canonical $\mathscr{O}_X$-module morphism $i : \mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee \to \mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ is a locally split embedding and there exists a unique canonical involution $\tau_0 : \mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \to \mathscr{C}_0(\mathscr{E}, q, \mathscr{L})^{\mathrm{op}}$ satisfying $\tau_0(i(v \otimes w \otimes f)) = i(w \otimes v \otimes f)$ for sections v, w of $\mathscr{E}$ and f of $\mathscr{L}^\vee$.

d) Any similarity $(\varphi, \lambda) : (\mathscr{E}, q, \mathscr{L}) \to (\mathscr{E}', q', \mathscr{L}')$ induces an $\mathscr{O}_X$-algebra isomorphism

$$\mathscr{C}_0(\varphi, \lambda) : \mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \to \mathscr{C}_0(\mathscr{E}', q', \mathscr{L}')$$

satisfying $i(v) \otimes i(w) \otimes f \mapsto i(\varphi(v)) \otimes i(\varphi(w)) \otimes f \circ \lambda^{-1}$ for sections v, w of $\mathscr{E}$ and f of $\mathscr{L}^\vee$.

e) Any quadratic alignment $A = (\mathscr{N}, \phi)$, with $\phi : \mathscr{N}^{\otimes 2} \otimes \mathscr{L} \to \mathscr{L}'$, induces an $\mathscr{O}_X$-algebra isomorphism

$$\mathscr{C}_0(A^{\circlearrowleft}) : \mathscr{C}_0(A^{\circlearrowleft}(\mathscr{E}, q, \mathscr{L})) \to \mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$$

satisfying $i(a \otimes v) \otimes i(b \otimes w) \otimes f \mapsto i(v) \otimes i(w) \otimes \phi'(a \otimes b \otimes f)$, for sections a, b of $\mathscr{N}$, v, w of $\mathscr{E}$, and f of $\mathscr{L}'^\vee$, where $\phi' : \mathscr{N}^{\otimes 2} \otimes \mathscr{L}'^\vee \to \mathscr{L}^\vee$ is the isomorphism canonically induced from ϕ.

f) For any morphism of schemes $p : X' \to X$, there is a canonical $\mathscr{O}_X$-module isomorphism

$$\mathscr{C}_0(p^*(\mathscr{E}, q, \mathscr{L})) \to p^* \mathscr{C}_0(\mathscr{E}, q, \mathscr{L}).$$

Proof. Properties *a* and *b* are étale local and hence follow from the corresponding properties of the classical even Clifford algebra (cf. [Knu2, IV Thm. 2.2.3, Prop. 3.2.4]), also see [BK, §3]. Properties *c*, *d*, and *e* are all consequence of the universal property. Property *f* is a direct consequence of the tensorial construction. □

Definition 1.3. Let $(\mathscr{E}, q, \mathscr{L})$ be a quadratic form of even rank on X. We call $f : Z = \mathbf{Spec}\, \mathscr{Z}(\mathscr{E}, q, \mathscr{L}) \to X$ the *discriminant cover* of $(\mathscr{E}, q, \mathscr{L})$. If $(\mathscr{E}, q, \mathscr{L})$ is regular, then $f : Z \to X$ is étale quadratic.

1.3. **Clifford bimodule.** As in the case of central simple algebras with orthogonal involution, line bundle-valued quadratic forms do not generally enjoy a “full” Clifford algebra, of which the even Clifford algebra is the even degree part. Inspired by [KMRT, II §9], we can directly define the *Clifford bimodule* $\mathscr{C}_1(\mathscr{E},q,\mathscr{L})$ of a quadratic form $(\mathscr{E},q,\mathscr{L})$, corresponding to the “odd” part of the classical Clifford algebra.

The $\mathscr{O}_X$-module $\mathscr{E}\otimes T(\mathscr{E}\otimes\mathscr{E}\otimes\mathscr{L}^\vee)$ has a natural right $T(\mathscr{E}\otimes\mathscr{E}\otimes\mathscr{L}^\vee)$-module structure denoted by $\otimes$. The $\mathscr{O}_X$-bilinear map $* : (\mathscr{E}\otimes\mathscr{E}\otimes\mathscr{L}^\vee)\times\mathscr{E}\to\mathscr{E}\otimes(\mathscr{E}\otimes\mathscr{E}\otimes\mathscr{L}^\vee)$ defined by

$$(u\otimes v\otimes f)*w = u\otimes(v\otimes w\otimes f)$$

for sections u,v,w of $\mathscr{E}$ and f of $\mathscr{L}^\vee$, induces a left $T(\mathscr{E}\otimes\mathscr{E}\otimes\mathscr{L}^\vee)$-module structure $*$ on $\mathscr{E}\otimes T(\mathscr{E}\otimes\mathscr{E}\otimes\mathscr{L}^\vee)$, uniquely defined so that it commutes with the natural right $T(\mathscr{E}\otimes\mathscr{E}\otimes\mathscr{L}^\vee)$-module structure. We define

$$\mathscr{C}_1(\mathscr{E},q,\mathscr{L}) = \mathscr{E}\otimes T(\mathscr{E}\otimes\mathscr{E}\otimes\mathscr{L}^\vee)/(\mathscr{E}\otimes\mathscr{J}_1+\mathscr{J}_1*\mathscr{E}) \tag{4}$$

together with the canonically induced morphism of $\mathscr{O}_X$-modules

$$i:\mathscr{E}\to\mathscr{C}_1(\mathscr{E},q,\mathscr{L}), \tag{5}$$

which is a locally split embedding. One immediately checks that $\mathscr{E}\otimes\mathscr{J}_2\subset\mathscr{J}_1*\mathscr{E}$ and $\mathscr{J}_2*\mathscr{E}\subset\mathscr{E}\otimes\mathscr{J}_1$, hence $\mathscr{C}_1(\mathscr{E},q,\mathscr{L})$ inherits a $\mathscr{C}_0(\mathscr{E},q,\mathscr{L})$-bimodule structure. Denote the right and left $\mathscr{C}_0(\mathscr{E},q,\mathscr{L})$-module structures by $\cdot$ and $*$, respectively.

Writing the rank as $n=2m$ or $n=2m+1$, there is a filtration

$$\mathscr{E}=\mathscr{F}_1\subset\mathscr{F}_3\subset\cdots\subset\mathscr{F}_{2m+1}=\mathscr{C}_1(\mathscr{E},q,\mathscr{L}),$$

where $\mathscr{F}_{2i+1}$ is the image of the truncation $\mathscr{E}\otimes T^{\le i}(\mathscr{E}\otimes\mathscr{E}\otimes\mathscr{L}^\vee)$ in $\mathscr{C}_1(\mathscr{E},q,\mathscr{L})$, for each $0\le i\le m$. This filtration has associated graded pieces $\mathscr{F}_{2i+1}/\mathscr{F}_{2i-1}\cong\bigwedge^{2i+1}\mathscr{E}\otimes(\mathscr{L}^\vee)^{\otimes i}$. In particular, $\mathscr{C}_1(\mathscr{E},q,\mathscr{L})$ is a locally free $\mathscr{O}_X$-module of rank 2^{n-1}. By its tensorial construction, the Clifford bimodule has the following.

Proposition 1.4 (Universal Property of the Clifford bimodule)**.** *Given a $\mathscr{C}_0(\mathscr{E},q,\mathscr{L})$-bimodule $\mathscr{B}$ (with right and left actions $\cdot$ and $*$) and an $\mathscr{O}_X$-module morphism $j:\mathscr{E}\to\mathscr{B}$ such that*

$$j(u)\cdot i(v\otimes w\otimes f) = i(u\otimes v\otimes f)*j(w),$$

for sections u,v,w of $\mathscr{E}$ and f of $\mathscr{L}^\vee$, there exists a unique $\mathscr{C}_0(\mathscr{E},q,\mathscr{L})$-bimodule morphism $\psi:\mathscr{C}_1(\mathscr{E},q,\mathscr{L})\to\mathscr{B}$ satisfying $j=\psi\circ i$.

The Clifford bimodule has the following additional properties.

Proposition 1.5. *Let $(\mathscr{E},q,\mathscr{L})$ be a regular quadratic form on a scheme X.*

- *a) The Clifford bimodule $\mathscr{C}_1(\mathscr{E},q,\mathscr{L})$ is invertible as a (left or right) $\mathscr{C}_0(\mathscr{E},q,\mathscr{L})$-module.*
- *b) If n is even, then the action of $\mathscr{Z}(\mathscr{E},q,\mathscr{L})$ on $\mathscr{C}_1(\mathscr{E},q,\mathscr{L})$ satisfies $x\cdot z=\iota(z)*x$ for sections z of $\mathscr{Z}(\mathscr{E},q,\mathscr{L})$ and x of $\mathscr{C}_1(\mathscr{E},q,\mathscr{L})$, where ι is the nontrivial $\mathscr{O}_X$-automorphism of $\mathscr{Z}(\mathscr{E},q,\mathscr{L})$.*
- *c) There is a canonical isomorphism*

  $$m:\mathscr{C}_1(\mathscr{E},q,\mathscr{L})\otimes_{\mathscr{C}_0(\mathscr{E},q,\mathscr{L})}\mathscr{C}_1(\mathscr{E},q,\mathscr{L})\to\mathscr{C}_0(\mathscr{E},q,\mathscr{L})\otimes_{\mathscr{O}_X}\mathscr{L}$$

  of $\mathscr{C}_0(\mathscr{E},q,\mathscr{L})$-bimodules satisfying $m(i(v)\otimes i(v))=1\otimes q(v)$ for a section v of $\mathscr{E}$.
- *d) Any similarity transformation $(\varphi,\lambda):(\mathscr{E},q,\mathscr{L})\to(\mathscr{E}',q',\mathscr{L}')$ induces an $\mathscr{O}_X$-module isomorphism*

  $$\mathscr{C}_1(\varphi,\lambda):\mathscr{C}_1(\mathscr{E},q,\mathscr{L})\to\mathscr{C}_1(\mathscr{E}',q',\mathscr{L}').$$

  that is $\mathscr{C}_0(\varphi,\lambda)$-semilinear with respect to the bimodule structure.

e) Any quadratic alignment $A = (\mathscr{N}, \phi)$, with $\phi : \mathscr{N}^{\otimes 2} \otimes \mathscr{L} \to \mathscr{L}'$, induces an $\mathscr{O}_X$-module isomorphism

$$\mathscr{C}_1(A^{\circlearrowleft}) : \mathscr{C}_1(A^{\circlearrowleft}(\mathscr{E}, q, \mathscr{L})) \to \mathscr{N} \otimes \mathscr{C}_1(\mathscr{E}, q, \mathscr{L})$$

that is $\mathscr{C}_0(A^{\circlearrowleft})$-semilinear with respect to the bimodule structure.

f) For any morphism of schemes $p : X' \to X$, there is a canonical $\mathscr{O}_X$-module isomorphism

$$\mathscr{C}_1(p^*(\mathscr{E}, q, \mathscr{L})) \to p^*\mathscr{C}_1(\mathscr{E}, q, \mathscr{L}).$$

Proof. For simplicity, we write $\mathscr{C}_0 = \mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ and $\mathscr{C}_1 = \mathscr{C}_1(\mathscr{E}, q, \mathscr{L})$. For *a*, since q is fiberwise nonzero, Zariski locally there exists a line subbundle $\mathscr{N} \subset \mathscr{E}$ such that $q|_{\mathscr{N}}$ is regular. Then as in the classical case (see [Knu2, IV Prop. 7.5.2]), $\mathscr{N}$ locally generates $\mathscr{C}_1$ over $\mathscr{C}_0$ as a right or left module.

For *b*, this is a local question and hence follows from [Knu2, IV Prop. 4.3.1(4)]. For *c*, we will define a $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$-bimodule morphism $\psi_m : \mathscr{C}_1(\mathscr{E}, q, \mathscr{L}) \to \mathscr{H}om_{\mathscr{C}_0}(\mathscr{C}_1, \mathscr{C}_0 \otimes \mathscr{L})$, where $\mathscr{H}om_{\mathscr{C}_0}$ denotes the sheaf of right $\mathscr{C}_0$-module homomorphisms (here $\mathscr{L}$ is acted trivially on). Then m will be the $\mathscr{C}_0$-bimodule map with adjoint ψ_m. To this end, for each section v of $\mathscr{E}$, we define a section m_v of $\mathscr{H}om_{\mathscr{C}_0}(\mathscr{C}_1, \mathscr{C}_0 \otimes \mathscr{L}) \cong \mathscr{H}om_{\mathscr{C}_0}(\mathscr{C}_1 \otimes \mathscr{L}^\vee \otimes \mathscr{L}, \mathscr{C}_0 \otimes \mathscr{L})$ by applying the universal property to the map $w \otimes f \otimes l \mapsto i(v \otimes w \otimes f) \otimes l$ for a section w of $\mathscr{E}$, f of $\mathscr{L}^\vee$, and l of $\mathscr{E}$. Then applying the universal property to the map defined by $v \mapsto m_v$, yields the required ψ_m. Finally, m is an isomorphism by *a*, since it's a nontrivial map of invertible $\mathscr{C}_0$-bimodules.

Properties *d* and *e* are consequence of the universal property (cf. [Bic, Prop. 2.6] and [BK, Lemma 3.3]). Property *f* is a direct consequence of the tensorial construction. □

1.4. **Metabolic forms.** A quadratic form $(\mathscr{E}, q, \mathscr{L})$ of rank $n = 2m$ on X is *metabolic* if there exists a locally direct summand $\mathscr{F} \to \mathscr{E}$ of rank m such that the restriction of q to $\mathscr{F}$ is zero. Any choice of such $\mathscr{P}$ is a *lagrangian*. The class of hyperbolic forms is the main example.

Example 1.6. For any vector bundle $\mathscr{P}$ of rank m and any line bundle $\mathscr{L}$ the *($\mathscr{L}$-valued) hyperbolic quadratic form* $H_{\mathscr{L}}(\mathscr{P})$ has underlying $\mathscr{O}_X$-module $\mathscr{H}om(\mathscr{P}, \mathscr{L}) \oplus \mathscr{P}$ and is given by $t + v \mapsto t(v)$ on sections. Here, $\mathscr{P}$ and $\mathscr{H}om(\mathscr{P}, \mathscr{L})$ are lagrangians.

We now proceed to compute the even Clifford algebra and Clifford bimodule of a hyperbolic form, which will be necessary for us later. Given an $\mathscr{O}_X$-module morphism $t : \mathscr{P} \to \mathscr{L}$, for each $i \geq 0$ we define

$$d_t^{(i)} : \textstyle\bigwedge^{i+1} \mathscr{P} \to \bigwedge^{i} \mathscr{P} \otimes \mathscr{L}$$

inductively by $d_t^{(i)}(v \wedge x) = x \otimes t(v) + x \wedge d_t^{(i-1)}(x)$ for sections v of $\mathscr{P}$ and x of $\bigwedge^i \mathscr{P}$, cf. [Auel2, §2]. Under the identification $\bigwedge^0 \mathscr{P} = \mathscr{O}_X$, we set $d_t^{(0)} = t$. Defining

$$\textstyle\bigwedge_{\mathscr{L}}^{+} \mathscr{P} = \displaystyle\bigoplus_{i=0}^{\lfloor m/2 \rfloor} \textstyle\bigwedge^{2i} \mathscr{P} \otimes (\mathscr{L}^\vee)^{\otimes i}, \qquad \textstyle\bigwedge_{\mathscr{L}}^{-} \mathscr{P} = \displaystyle\bigoplus_{i=0}^{\lfloor (m-1)/2 \rfloor} \textstyle\bigwedge^{2i+1} \mathscr{P} \otimes (\mathscr{L}^\vee)^{\otimes i}.$$

there are induced $\mathscr{O}_X$-module morphisms

$$d_t^{+} : \textstyle\bigwedge_{\mathscr{L}}^{+} \mathscr{P} \to \bigwedge_{\mathscr{L}}^{-} \mathscr{P}, \qquad d_t^{-} : \bigwedge_{\mathscr{L}}^{-} \mathscr{P} \to \bigwedge_{\mathscr{L}}^{+} \mathscr{P} \otimes \mathscr{L}.$$

Also, for each global section v of $\mathscr{P}$, left wedging defines $\mathscr{O}_X$-module morphisms

$$l_v^{+} : \textstyle\bigwedge_{\mathscr{L}}^{+} \mathscr{P} \to \bigwedge_{\mathscr{L}}^{-} \mathscr{P}, \qquad l_v^{-} : \bigwedge_{\mathscr{L}}^{-} \mathscr{P} \to \bigwedge_{\mathscr{L}}^{+} \mathscr{P} \otimes \mathscr{L}.$$

One immediately checks that the maps

$$\begin{aligned} H_{\mathscr{L}}(\mathscr{P}) \otimes H_{\mathscr{L}}(\mathscr{P}) \otimes \mathscr{L}^\vee &\to \mathscr{E}nd\left(\textstyle\bigwedge_{\mathscr{L}}^{+} \mathscr{P}\right) \times \mathscr{E}nd\left(\textstyle\bigwedge_{\mathscr{L}}^{-} \mathscr{P}\right) \\ (t+v) \otimes (s+w) \otimes f &\mapsto (\mathrm{id} \otimes f)(d_t^{-} \circ d_s^{+} + d_t^{-} \circ l_w^{+} + l_v^{-} \circ d_s^{+} + l_v^{-} \circ l_w^{+}) \\ &\qquad + (d_t^{+} \otimes f \circ d_s^{-} + d_t^{+} \otimes f \circ l_w^{-} + l_v^{+} \otimes f \circ d_s^{-} + l_v^{+} \otimes f \circ l_w^{-}) \end{aligned}$$

and

$$\begin{aligned} H_{\mathscr{L}}(\mathscr{P}) &\to \mathscr{H}om\big(\textstyle\bigwedge^+_{\mathscr{L}}\mathscr{P}, \bigwedge^-_{\mathscr{L}}\mathscr{P}\big) \oplus \mathscr{H}om\big(\bigwedge^-_{\mathscr{L}}\mathscr{P}, \bigwedge^+_{\mathscr{L}}\mathscr{P}\big) \otimes \mathscr{L} \\ t + v &\mapsto (d_t^+ + l_v^+) + (d_t^- + l_v^-). \end{aligned}$$

satisfy the universal properties of the even Clifford algebra and Clifford bimodule, hence induce a canonical $\mathscr{O}_X$-algebra morphism

$$\Phi_0 : \mathscr{C}_0(H_{\mathscr{L}}(\mathscr{P})) \to \mathscr{E}nd\left(\textstyle\bigwedge^+_{\mathscr{L}}\mathscr{P}\right) \times \mathscr{E}nd\left(\bigwedge^-_{\mathscr{L}}\mathscr{P}\right)$$

and a canonical $\mathscr{O}_X$-module morphism

$$\Phi_1 : \mathscr{C}_1(H_{\mathscr{L}}(\mathscr{P})) \to \mathscr{H}om\big(\textstyle\bigwedge^+_{\mathscr{L}}\mathscr{P}, \bigwedge^-_{\mathscr{L}}\mathscr{P}\big) \oplus \mathscr{H}om\big(\bigwedge^-_{\mathscr{L}}\mathscr{P}, \bigwedge^+_{\mathscr{L}}\mathscr{P}\big) \otimes \mathscr{L}$$

transporting, via the morphism Φ_0, the $\mathscr{C}_0(H_{\mathscr{L}}(\mathscr{P}))$-bimodule structure to the evident composition $\mathscr{E}nd(\bigwedge^+_{\mathscr{L}}\mathscr{P}) \times \mathscr{E}nd(\bigwedge^-_{\mathscr{L}}\mathscr{P})$-bimodule structure. Zariski locally, Φ_0 and Φ_1 agree with the restriction of the classical isomorphism $\mathscr{C}(H_{\mathscr{O}_X}(\mathscr{P})) \cong \mathscr{E}nd(\bigwedge \mathscr{P})$ (see [Knu2, IV Prop. 2.1.1]) to the even and odd components of the Clifford algebra, hence Φ_0 and Φ_1 are isomorphisms.

We point out that $\mathscr{Z}(H_{\mathscr{L}}(\mathscr{P})) \cong \mathscr{O}_X \times \mathscr{O}_X$ is the split étale quadratic algebra.

The formula for the even Clifford algebra of a hyperbolic form given in Example 1.6 does not persist to (nonsplit) metabolic quadratic forms, a phenomenon already apparent when $\mathscr{L} = \mathscr{O}_X$; see [KO]. However, the main result of this section is that $\mathscr{C}_0$ is still a product of split Azumaya algebras.

Theorem 1.7. *Let $(\mathscr{E}, q, \mathscr{L})$ be a metabolic quadratic form of rank $n = 2m$ on a scheme X. Any choice of lagrangian $\mathscr{F} \to \mathscr{E}$ induces a natural choice of vector bundles $\mathscr{M}^+$ and $\mathscr{M}^-$ of rank 2^{m-1}, an $\mathscr{O}_X$-algebra isomorphism*

$$\Phi_0 : \mathscr{C}_0(\mathscr{E}, q, \mathscr{L}) \cong \mathscr{E}nd(\mathscr{M}^+) \times \mathscr{E}nd(\mathscr{M}^-),$$

and an $\mathscr{O}_X$-module isomorphism

$$\Phi_1 : \mathscr{C}_1(\mathscr{E}, q, \mathscr{L}) \cong \mathscr{H}om(\mathscr{M}^+, \mathscr{M}^-) \oplus \mathscr{H}om(\mathscr{M}^-, \mathscr{M}^+) \otimes \mathscr{L}$$

transporting, via Φ_0, the $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$-bimodule structure to the evident composition $\mathscr{E}nd(\mathscr{M}^+) \times \mathscr{E}nd(\mathscr{M}^-)$-bimodule structure.

Proof. We generalize the proof from Knus–Ojanguren [KO] to the line bundle-valued setting. On the category of vector bundles, write $(-)^{\vee\mathscr{L}}$ for the functor $\mathscr{H}om(-, \mathscr{L})$ and $\mathrm{can}_{\mathscr{L}}$ for the canonical isomorphism of functors $\mathrm{id} \to ((-)^{\vee\mathscr{L}})^{\vee\mathscr{L}}$.

Let $\mathscr{P}$ be a vector bundle of rank $m \geq 1$ and $H_{\mathscr{L}}(\mathscr{P})$ be the corresponding $\mathscr{L}$-valued hyperbolic form. Denote by $\gamma_0 : \mathbf{O}\big(H_{\mathscr{L}}(\mathscr{P})\big) \to \mathbf{Aut}_{\mathscr{O}_X\text{-alg}}\big(\mathscr{C}_0(H_{\mathscr{L}}(\mathscr{P}))\big)$ the homomorphism induced by Proposition 1.2*d*. Restricting γ_0 to the center yields the *Dickson* homomorphism $\Delta : \mathbf{O}\big(H_{\mathscr{L}}(\mathscr{P})\big) \to \mathbf{Aut}_{\mathscr{O}_X\text{-alg}}\big(\mathscr{Z}(H_{\mathscr{L}}(\mathscr{P}))\big) = \mathbb{Z}/2\mathbb{Z}$ of group schemes, cf. [Auel1, §1.9]. Its kernel is the special orthogonal group scheme $\mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P})\big)$. Under the identification $\mathscr{C}_0(H_{\mathscr{L}}(\mathscr{P})) = \mathscr{E}nd\big(\bigwedge^+_{\mathscr{L}}\mathscr{P}\big) \times \mathscr{E}nd\big(\bigwedge^-_{\mathscr{L}}\mathscr{P}\big)$ of Example 1.6, we have that γ_0 restricts to a homomorphism

$$\gamma_0 : \mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P})\big) \to \mathbf{Aut}_{\mathscr{Z}\text{-alg}}\big(\mathscr{C}_0(H_{\mathscr{L}}(\mathscr{P}))\big) \cong \mathbf{PGL}\big(\textstyle\bigwedge^+_{\mathscr{L}}\mathscr{P}\big) \times \mathbf{PGL}\big(\bigwedge^-_{\mathscr{L}}\mathscr{P}\big).$$

Similarly, denote by $\gamma_1 : \mathbf{O}\big(H_{\mathscr{L}}(\mathscr{P})\big) \to \mathbf{Aut}_{\mathscr{O}_X-\mathrm{mod}}\big(\mathscr{C}_1(H_{\mathscr{L}}(\mathscr{P}))\big)$ the homomorphism induced by Proposition 1.5*d*. Under the identification of $\mathscr{C}_1(H_{\mathscr{L}}(\mathscr{P}))$ with the vector bundle $\mathscr{H}om\left(\bigwedge^+_{\mathscr{L}}\mathscr{P}, \bigwedge^-_{\mathscr{L}}\mathscr{P}\right) \oplus \mathscr{H}om\left(\bigwedge^-_{\mathscr{L}}\mathscr{P}, \bigwedge^+_{\mathscr{L}}\mathscr{P}\right) \otimes \mathscr{L}$ of Example 1.6, we have that γ_1 restricts to a homomorphism

$$\gamma_1 : \mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P})\big) \to \mathbf{Aut}_{\mathscr{Z}}\big(\mathscr{C}_1(H_{\mathscr{L}}(\mathscr{P}))\big) \cong \mathbf{GL}(\mathscr{H}^+) \times \mathbf{GL}(\mathscr{H}^-)$$

where we write $\mathscr{H}^+ = \mathscr{H}om\big(\bigwedge^+_{\mathscr{L}}\mathscr{P}, \bigwedge^-_{\mathscr{L}}\mathscr{P}\big)$ and $\mathscr{H}^- = \mathscr{H}om\big(\bigwedge^-_{\mathscr{L}}\mathscr{P}, \bigwedge^+_{\mathscr{L}}\mathscr{P} \otimes \mathscr{L}\big)$.

The parabolic subgroup $\mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P}),\mathscr{P}\big) \subset \mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P})\big)$ of isometries preserving $\mathscr{P}$ has the following block description

$$\mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P}),\mathscr{P}\big)(U) = \left\{ \begin{pmatrix} (\alpha^{\vee\mathscr{L}})^{-1} & \beta \\ 0 & \alpha \end{pmatrix} \ : \ \beta^{\vee\mathscr{L}}\,\mathrm{can}_{\mathscr{L}}\,\alpha \text{ is alternating} \right\}$$

where for each $U \to X$ and each $\alpha \in \mathrm{Hom}(\mathscr{P}|_U,\mathscr{P}|_U)$ and $\beta \in \mathrm{Hom}(\mathscr{P}|_U,\mathscr{P}|_U^{\vee\mathscr{L}})$, we consider $\beta^{\vee\mathscr{L}}\,\mathrm{can}_{\mathscr{L}}\,\alpha : \mathscr{P}_U \to \mathscr{P}|_U^{\vee\mathscr{L}}$ as the adjoint of an $\mathscr{L}|_U$-valued bilinear form.

We use an $\mathscr{L}$-valued version of Bourbaki's tensor operations for even Clifford algebras, cf. [Bal, Thm. 2.2]. In particular, under the canonical identifications $\mathscr{C}_0(\mathscr{P},0,\mathscr{L}) = \bigwedge_{\mathscr{L}}^{+}\mathscr{P}$ and $\mathscr{C}_1(\mathscr{P},0,\mathscr{L}) = \bigwedge_{\mathscr{L}}^{-}\mathscr{P}$, there exist homomorphisms of sheaves of groups

$$\Psi^{\pm} : \mathscr{H}om\big(\textstyle\bigwedge^2\mathscr{P},\mathscr{L}\big) \to \mathbf{GL}\big(\textstyle\bigwedge_{\mathscr{L}}^{\pm}\mathscr{P}\big)$$

satisfying the following properties:

$$\Psi^{\pm}(b\circ\wedge^2\varphi) = \wedge_{\mathscr{L}}^{\pm}(\varphi)^{-1}\ \Psi^{\pm}(b)\ \wedge_{\mathscr{L}}^{\pm}(\varphi) \tag{6}$$

for each alternating form $b : \bigwedge^2\mathscr{P} \to \mathscr{L}$ and each $\varphi \in \mathbf{GL}(\mathscr{P})$; and

$$\psi_b = \psi_{b'} \quad \Rightarrow \quad \Psi^{\pm}(b) = \Psi^{\pm}(b') \tag{7}$$

where $\psi_b : \mathscr{P} \to \mathscr{P}^{\vee\mathscr{L}}$ is the adjoint map to the alternating form $b : \bigwedge^2\mathscr{P} \to \mathscr{L}$. By (7), we can write $\Psi^{\pm}(\psi)$ in place of $\Psi^{\pm}(b)$ for any $\mathscr{O}_X$-module morphism $\psi : \mathscr{P} \to \mathscr{P}^{\vee\mathscr{L}}$ that is adjoint to an alternating form $b : \bigwedge^2\mathscr{P} \to \mathscr{L}$.

With this in hand, we define maps

$$\begin{aligned} \rho^{\pm} : \mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P}),\mathscr{P}\big) &\to \mathbf{GL}\big(\textstyle\bigwedge_{\mathscr{L}}^{\pm}\mathscr{P}\big) \\ \begin{pmatrix} (\alpha^{\vee\mathscr{L}})^{-1} & \beta \\ 0 & \alpha \end{pmatrix} &\mapsto \wedge_{\mathscr{L}}^{\pm}(\alpha)\,\Psi^{\pm}(\alpha^{\vee\mathscr{L}}\beta) \end{aligned}$$

which we now proceed to verify are well defined homomorphisms. Consider the Levi decomposition $\mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P}),\mathscr{P}\big) = \mathbf{MN} = \mathbf{NM}$ given explicitly by

$$\begin{pmatrix} (\alpha^{\vee\mathscr{L}})^{-1} & \beta \\ 0 & \alpha \end{pmatrix} = \begin{pmatrix} (\alpha^{\vee\mathscr{L}})^{-1} & 0 \\ 0 & \alpha \end{pmatrix}\begin{pmatrix} 1 & \alpha^{\vee\mathscr{L}}\beta \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \beta\alpha^{-1} \\ 0 & 1 \end{pmatrix}\begin{pmatrix} (\alpha^{\vee\mathscr{L}})^{-1} & 0 \\ 0 & \alpha \end{pmatrix}$$

and note that $\alpha^{\vee\mathscr{L}}\beta$ (being the transpose of $\beta^{\vee\mathscr{L}}\,\mathrm{can}_{\mathscr{L}}\,\alpha$) is adjoint to an alternating form, say $b : \bigwedge^2\mathscr{P} \to \mathscr{L}$. Then $\beta\alpha^{-1}$ is adjoint to the alternating form $b\circ\wedge^2\alpha^{-1}$, since we can write $\beta\alpha^{-1} = (\alpha^{-1})^{\vee\mathscr{L}}(\alpha^{\vee\mathscr{L}}\beta)\alpha^{-1}$. Hence by (6), $\rho^{\pm}$ is also given by $\Psi^{\pm}(\beta\alpha^{-1})\wedge_{\mathscr{L}}^{\pm}(\alpha)$. Since $\rho^{\pm}$ is based on, and independent of, the Levi decomposition order, it is a well defined group scheme homomorphism.

Denoting by $\rho_0 = \rho^+ \times \rho^- : \mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P}),\mathscr{P}\big) \to \mathbf{GL}\big(\bigwedge_{\mathscr{L}}^{+}\mathscr{P}\big) \times \mathbf{GL}\big(\bigwedge_{\mathscr{L}}^{-}\mathscr{P}\big)$, consider the diagram

$$\begin{tikzcd} \mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P}),\mathscr{P}\big) \arrow[r] \arrow[d, "\rho_0"'] & \mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P})\big) \arrow[d, "\gamma_0"] \\ \mathbf{GL}\big(\bigwedge_{\mathscr{L}}^{+}\mathscr{P}\big) \times \mathbf{GL}\big(\bigwedge_{\mathscr{L}}^{-}\mathscr{P}\big) \arrow[r] & \mathbf{PGL}\big(\bigwedge_{\mathscr{L}}^{+}\mathscr{P}\big) \times \mathbf{PGL}\big(\bigwedge_{\mathscr{L}}^{-}\mathscr{P}\big) \end{tikzcd}$$

of group schemes, where the horizontal arrows are the obvious ones. The fiber of this diagram over any point of X is isomorphic to the restriction, to the special orthogonal group and even Clifford algebra, of the corresponding commutative diagram of orthogonal groups and (full) Clifford algebras in [KO, Thm.] (cf. [Knu2, IV Prop. 2.4.2]). Hence the diagram commutes over X.

We now consider the induced commutative diagram of pointed nonabelian cohomology sets: $H^1_{\text{ét}}\big(X,\mathbf{SO}(H_{\mathscr{L}}(\mathscr{P}))\big)$ is in bijection with the set of similarity classes of $\mathscr{L}$-valued quadratic forms $(\mathscr{E},q,\mathscr{L})$ of rank $2m$ together with an *orientation* isomorphism $\zeta : \mathscr{Z}(\mathscr{E},q,\mathscr{L}) \cong \mathscr{O}_X \times$

$\mathscr{O}_X$ (cf. [Auel1, Prop. 1.15]); $H^1_{\text{ét}}(X, \mathbf{SO}(H_{\mathscr{L}}(\mathscr{P}), \mathscr{P}))$ is in bijection with the set of similarity classes of metabolic $\mathscr{L}$-valued quadratic forms $(\mathscr{E}, q, \mathscr{L})$ of rank $n = 2m$ together with a choice of lagrangian; $H^1_{\text{ét}}(X, \mathbf{GL}(\bigwedge^{\pm}_{\mathscr{L}} \mathscr{P}))$ is in bijection with the set of isomorphism classes of vector bundles $\mathscr{M}^{\pm}$ of rank 2^{m-1}; $H^1_{\text{ét}}(X, \mathbf{PGL}(\bigwedge^{\pm}_{\mathscr{L}} \mathscr{P}))$ is in bijection with the set of isomorphism classes of Azumaya algebras of degree 2^{m-1}; the induced map

$$H^1_{\text{ét}}\big(X, \mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P}), \mathscr{P}\big)\big) \to H^1_{\text{ét}}\big(X, \mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P})\big)\big)$$

replaces the choice of lagrangian by the orientation it canonically induces (cf. [Auel1, Lemma 1.14]); the map induced by γ_0 takes an oriented quadratic form to its even Clifford algebra together with the splitting of its center induced by the orientation; the map induced by ρ_0 takes a metabolic quadratic form of rank $n = 2m$ together with a choice of lagrangian to a pair of vector bundles $\mathscr{M}^+$ and $\mathscr{M}^-$ of rank 2^{m-1}; the induced map

$$H^1_{\text{ét}}\big(X, \mathbf{GL}\big(\textstyle\bigwedge^{\pm}_{\mathscr{L}} \mathscr{P}\big)\big) \to H^1_{\text{ét}}\big(X, \mathbf{PGL}\big(\textstyle\bigwedge^{\pm}_{\mathscr{L}} \mathscr{P}\big)\big)$$

takes a vector bundle $\mathscr{M}^{\pm}$ to the Azumaya algebra $\mathscr{E}nd(\mathscr{M}^{\pm})$. Chasing the diagram around shows that if $(\mathscr{E}, q, \mathscr{L})$ is a metabolic quadratic form of rank $2m$, then $\mathscr{C}_0(\mathscr{E}, b, \mathscr{L})$ is isomorphic to $\mathscr{E}nd(\mathscr{M}^+) \times \mathscr{E}nd(\mathscr{M}^-)$ for vector bundles $\mathscr{M}^+$ and $\mathscr{M}^-$ of rank 2^{m-1} on X.

To identify the Clifford bimodule, consider the diagram

$$\begin{tikzcd}
\mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P}), \mathscr{P}\big) \arrow[rr] \arrow[d, "\rho_0"'] & & \mathbf{SO}\big(H_{\mathscr{L}}(\mathscr{P})\big) \arrow[d, "\gamma_1"] \\
\mathbf{GL}\big(\bigwedge^{+}_{\mathscr{L}} \mathscr{P}\big) \times \mathbf{GL}\big(\bigwedge^{-}_{\mathscr{L}} \mathscr{P}\big) \arrow[rr, "c"] & & \mathbf{GL}(\mathscr{H}^+) \times \mathbf{GL}(\mathscr{H}^-)
\end{tikzcd}$$

of group schemes, where the top horizontal arrow is the canonical one, and c is the evident homomorphism defined by compositions. The fiber of this diagram over any point of X is isomorphic to the restriction, to the special orthogonal group and odd part of the Clifford algebra, of the corresponding commutative diagram of orthogonal groups and (full) Clifford algebras in [KO, Thm.] (cf. [Knu2, IV Prop. 2.4.2]). Hence the diagram commutes over X.

As above, we consider the induced commutative diagram of pointed nonabelian cohomology sets: the map induced by γ_1 takes an oriented quadratic form to its Clifford bimodule, together with a direct sum decomposition stable under the action of the center; the map induced by c takes a pair of vector bundles $\mathscr{M}^+$ and $\mathscr{M}^-$ of rank 2^{m-1} to $\mathscr{H}om(\mathscr{M}^+, \mathscr{M}^-) \oplus \mathscr{H}om(\mathscr{M}^-, \mathscr{M}^+ \otimes \mathscr{L})$. Chasing the diagram around gives the stated identification. The compatibility of the bimodule structures can then be checked locally. $\square$

1.5. **Orthogonal sums.** We will also need an orthogonal sum formula for the even Clifford algebra. Let $(\mathscr{E}, q, \mathscr{L})$ and $(\mathscr{E}', q', \mathscr{L})$ be quadratic forms over a scheme X and denote by

$$i_0 : \mathscr{E} \otimes \mathscr{E} \otimes \mathscr{L}^\vee \to \mathscr{C}_0(q), \qquad i_0' : \mathscr{E}' \otimes \mathscr{E}' \otimes \mathscr{L}^\vee \to \mathscr{C}_0(q'),$$
$$i_1 : \mathscr{E} \to \mathscr{C}_1(q), \qquad i_1' : \mathscr{E}' \to \mathscr{C}_1(q'),$$

the canonical $\mathscr{O}_X$-module morphisms (3) and (5), respectively.

We define an $\mathscr{O}_X$-algebra structure on $\mathscr{C}_0(q) \otimes \mathscr{C}_0(q') \oplus \mathscr{C}_1(q) \otimes \mathscr{C}_1(q') \otimes \mathscr{L}^\vee$ as follows: by multiplication in $\mathscr{C}_0$ (for products between elements of the first summand), by the $\mathscr{C}_0$-bimodule action $\mathscr{C}_1$ (between elements of the first and second summands), and by the multiplication map $m : \mathscr{C}_1 \otimes \mathscr{C}_1 \to \mathscr{C}_0 \otimes \mathscr{L}$ in Proposition 1.5c (between elements of the second summand) followed by evaluation with $\mathscr{L}^\vee$. One can check that the map

$$\begin{aligned}
(\mathscr{E} \oplus \mathscr{E}') \otimes (\mathscr{E} \oplus \mathscr{E}') \otimes \mathscr{L}^\vee &\to \mathscr{C}_0(q) \otimes \mathscr{C}_0(q') \oplus \mathscr{C}_1(q) \otimes \mathscr{C}_1(q') \otimes \mathscr{L}^\vee \\
(v + v') \otimes (w + w') \otimes f &\mapsto \big(i_0(v \otimes w \otimes f) \otimes 1 + 1 \otimes i_0'(v' \otimes w' \otimes f)\big) \\
&\qquad + \big(i_1(v) \otimes i_1'(w') \otimes f - i_1(w) \otimes i_1'(v') \otimes f\big)
\end{aligned}$$

satisfies the universal property of the even Clifford algebra, hence induces an $\mathscr{O}_X$-algebra morphism $\mathscr{C}_0(q \perp q') \to \mathscr{C}_0(q) \otimes \mathscr{C}_0(q') \oplus \mathscr{C}_1(q) \otimes \mathscr{C}_1(q') \otimes \mathscr{L}^\vee$. Via this morphism, there is an induced $\mathscr{C}_0(q \perp q')$-bimodule structure on $\mathscr{C}_0(q) \otimes \mathscr{C}_1(q') \oplus \mathscr{C}_1(q) \otimes \mathscr{C}_0(q')$, and one can check that the map

$$\begin{aligned} \mathscr{E} \oplus \mathscr{E}' &\to \mathscr{C}_0(q) \otimes \mathscr{C}_1(q') \oplus \mathscr{C}_1(q) \otimes \mathscr{C}_0(q') \\ v + v' &\mapsto i_1(v) \otimes 1 + 1 \otimes i_1'(v') \end{aligned}$$

satisfies the universal property of the Clifford bimodule.

Theorem 1.8. *Let $(\mathscr{E}, q, \mathscr{L})$ and $(\mathscr{E}', q', \mathscr{L})$ be quadratic forms over a scheme X. Then the $\mathscr{O}_X$-algebra morphism*

$$\mathscr{C}_0(q \perp q') \to \mathscr{C}_0(q) \otimes \mathscr{C}_0(q') \oplus \mathscr{C}_1(q) \otimes \mathscr{C}_1(q') \otimes \mathscr{L}^\vee \tag{8}$$

and the $\mathscr{C}_0(q \perp q')$-bimodule morphism

$$\mathscr{C}_1(q \perp q') \to \mathscr{C}_0(q) \otimes \mathscr{C}_1(q') \oplus \mathscr{C}_1(q) \otimes \mathscr{C}_0(q'), \tag{9}$$

induced from the universal properties, are isomorphisms.

Proof. Locally, when $\mathscr{L}$ is trivial, these maps agree with their classical counterparts (cf. [Knu2, IV Thm. 1.3.1]) and hence are isomorphisms. $\square$

2. Total Witt groups and total classical invariants

In this section, we define the notion of total Witt groups and construct the total classical cohomological invariants on these groups.

2.1. **Total Witt groups.** One must be careful when working with "total" Witt groups. Fix a scheme X and denote by $W(X, \mathscr{L})$ the (quadratic) Witt group of regular $\mathscr{L}$-valued quadratic forms modulo metabolic forms on X. We usually write $W(X) = W(X, \mathscr{O}_X)$. Every Witt class can be represented by a regular quadratic form, see [Kneb2, §5].

We also fix a set P of line bundle representatives of the quotient group $\mathrm{Pic}(X)/2$. With respect to this choice, we define the *total (quadratic) Witt group* $W_{\mathrm{tot}}(X) = \bigoplus_{\mathscr{L} \in P} W(X, \mathscr{L})$. While the abelian group $W_{\mathrm{tot}}(X)$ is only well defined up to non-canonical isomorphism depending on our choice of P, the cohomological invariants we consider will not depend on such choices. Definition 2.1 makes this precise.

Most importantly, we will not consider any ring structure on $W_{\mathrm{tot}}(X)$ and thus will not need to descend into the subtle considerations of [BC].

Definition 2.1. Fix an abelian group H and group homomorphisms $e_{\mathscr{L}} : W(X, \mathscr{L}) \to H$ for each line bundle $\mathscr{L}$. We say that the system $\{e_{\mathscr{L}}\}$ is a *system of projective similarity class invariants* if for any quadratic alignment $A = (\mathscr{N}, \phi)$ between line bundles $\mathscr{L}$ and $\mathscr{L}'$, there is a commutative diagram

$$\begin{tikzcd} W(X, \mathscr{L}) \arrow[r, "e_{\mathscr{L}}"] \arrow[d, "A^{\circlearrowleft}"'] & H \arrow[d, equal] \\ W(X, \mathscr{L}') \arrow[r, "e_{\mathscr{L}'}"] & H \end{tikzcd}$$

of abelian groups. One could axiomatize this notion using the language of morphisms of functors, together with a compatibility condition with respect to quadratic alignments.

Given a system $\{e_{\mathscr{L}}\}$ of projective similarity class invariants, the combined homomorphism

$$e = \oplus_{\mathscr{L} \in P}\, e_{\mathscr{L}} : W_{\mathrm{tot}}(X) = \bigoplus\nolimits_{\mathscr{L} \in P} W(X, \mathscr{L}) \to H$$

is well defined and independent of the choice P of representatives of $\mathrm{Pic}(X)/2$. We call e the *total invariant* associated to the system $\{e_{\mathscr{L}}\}$ of projective similarity class invariants.

A reader who is unhappy with this formalism may, for example, simply replace the statement "$e : W_{\mathrm{tot}}(X) \to H$ is surjective" by the equivalent statement "for each $h \in H$, there exists a line bundle $\mathscr{L}$ and a class $q \in W(X, \mathscr{L})$, such that $e_{\mathscr{L}}(q) = h$".

2.2. **Rank modulo 2.** For each line bundle $\mathscr{L}$, the rank modulo 2 defines a homomorphism $e^0_{\mathscr{L}} : W(X, \mathscr{L}) \to H^0_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$. Then $\{e^0_{\mathscr{L}}\}$ is a system of projective similarity class invariants and there is a *total rank modulo 2* homomorphism

$$e^0 : W_{\mathrm{tot}}(X) \to H^0_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z}).$$

Denote by $I^1(X, \mathscr{L}) \subset W(X, \mathscr{L})$ the kernel of $e^0_{\mathscr{L}}$ and by $I^1_{\mathrm{tot}}(X) = \bigoplus_{\mathscr{L} \in P} I^1(X, \mathscr{L})$. Note that if $\mathscr{L}$ is not a square in $\mathrm{Pic}(X)$ then $I^1(X, \mathscr{L}) = W(X, \mathscr{L})$, cf. [Auel1, Lemma 1.6]. Thus e^0 has kernel $I^1_{\mathrm{tot}}(X)$.

2.3. **Total discriminant.** Recall from Proposition 1.2*a* that the center $\mathscr{Z}(\mathscr{E}, q, \mathscr{L})$ of the even Clifford algebra $\mathscr{C}_0(\mathscr{E}, q, \mathscr{L})$ of a regular quadratic form $(\mathscr{E}, q, \mathscr{L})$ is an étale quadratic $\mathscr{O}_X$-algebra. We call its X-algebra isomorphism class in $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ the *discriminant invariant* $d(\mathscr{E}, q, \mathscr{L})$.

Remark 2.2. If 2 is invertible on X and $(\mathscr{E}, q, \mathscr{L})$ is a regular quadratic form of even rank $n = 2m$, then under the canonical homomorphism $H^1(X, \mathbb{Z}/2\mathbb{Z}) \to H^1(X, \boldsymbol{\mu}_2)$, the discriminant invariant $d(\mathscr{E}, q, \mathscr{L})$ maps to the class of the signed discriminant module $\det \mathscr{E} \otimes (\mathscr{L}^\vee)^{\otimes m}$ (defined in [PS2, §4]) of the associated symmetric bilinear form b_q, cf. [Auel1, §1.9].

Proposition 2.3. *Let $(\mathscr{E}, q, \mathscr{L})$ and $(\mathscr{E}', q', \mathscr{L})$ be regular quadratic forms of even rank over a scheme X. Then $d(q \perp q') = d(q) + d(q')$ in $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$.*

Proof. We recall (cf. [Knu2, III Prop. 4.1.4]) that given étale quadratic $\mathscr{O}_X$-algebras $\mathscr{Z}$ and $\mathscr{Z}'$, the addition of classes $[\mathscr{Z}]$ and $[\mathscr{Z}']$ in $H^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$ is represented by the quadratic étale algebra $\mathscr{Z} \circ \mathscr{Z}'$, defined to be the $\mathscr{O}_X$-subalgebra of $\mathscr{Z} \otimes \mathscr{Z}'$ invariant under the diagonal Galois action $\iota \otimes \iota'$, where ι and ι' are the nontrivial $\mathscr{O}_X$-automorphisms of $\mathscr{Z}$ and $\mathscr{Z}'$, respectively.

Using Proposition 1.5*b*, we see that restricting the isomorphism (8) to the center yields an $\mathscr{O}_X$-algebra morphism $\mathscr{Z}(q \perp q') \to \mathscr{Z}(q) \otimes \mathscr{Z}(q')$, which we claim factors through $\mathscr{Z}(q) \circ \mathscr{Z}(q')$. Indeed, for any section $v \otimes v' \otimes f$ of $\mathscr{E}_1 \otimes \mathscr{E}_2$ and $z \otimes z'$ of $\mathscr{Z}(q) \otimes \mathscr{Z}(q')$, we have that

$$(v \otimes v' \otimes f)\,(z \otimes z') = (v \cdot z) \otimes (v' \cdot z') \otimes f = (\iota(z) * v) \otimes (\iota'(z') * v') \otimes f = (\iota \otimes \iota')(z \otimes z')\,(v \otimes v' \otimes f)$$

by Proposition 1.5*b*,*c*, where we suppress the canonical embeddings (3), (5) and the isomorphism (8). Hence $(\mathrm{id} - \iota \otimes \iota')\mathscr{Z}(q \perp q')$ annihilates $\mathscr{E} \otimes \mathscr{E}' \otimes \mathscr{L}^\vee$, hence is zero. This proves the claim. The resulting $\mathscr{O}_X$-algebra morphism $\mathscr{Z}(q \perp q') \to \mathscr{Z}(q) \circ \mathscr{Z}(q')$ is Zariski locally an isomorphism by [Knu2, IV §4.4], hence is an isomorphism. □

As a consequence, for each line bundle $\mathscr{L}$, the discriminant invariant defines a homomorphism $e^1_{\mathscr{L}} : I^1(X, \mathscr{L}) \to H^1_{\mathrm{fppf}}(X, \mathbb{Z}/2\mathbb{Z})$. By Proposition 1.2*e*, $\{e^1_{\mathscr{L}}\}$ is a system of projective similarity class invariants and there is a *total discriminant invariant* homomorphism

$$e^1 : I^1_{\mathrm{tot}}(X) \to H^1_{\mathrm{fppf}}(X, \mathbb{Z}/2\mathbb{Z}).$$

Denote by $I^2(X, \mathscr{L}) \subset I^1(X, \mathscr{L})$ the kernel of $e^1_{\mathscr{L}}$ and by $I^2_{\mathrm{tot}}(X) = \oplus_{\mathscr{L} \in P} I^2(X, \mathscr{L})$. Then $I^2_{\mathrm{tot}}(X) \subset \ker(e^1)$.

Remark 2.4. The quotient group $\ker(e^1)/I^2_{\mathrm{tot}}(X)$ is generated by elements of the form $[\mathscr{E}, q, \mathscr{L}] - [\mathscr{E}', q', \mathscr{L}']$, where both q and q' have equal *nontrivial* discriminant invariant and yet $\mathscr{L}$ and $\mathscr{L}'$ are in different square classes. This group will be the subject of future investigation.

2.4. **Total Clifford invariant.** For a regular quadratic form $(\mathscr{E},q,\mathscr{L})$ of even rank $n=2m$ and trivial discriminant on X, the even Clifford algebra decomposes as a product of Azumaya $\mathscr{O}_X$-algebras $\mathscr{C}_0(\mathscr{E},q,\mathscr{L})\cong\mathscr{C}_0^+(\mathscr{E},q,\mathscr{L})\times\mathscr{C}_0^-(\mathscr{E},q,\mathscr{L})$ upon fixing a splitting idempotent of the center $\mathscr{Z}(\mathscr{E},q,\mathscr{L})\cong\mathscr{O}_X\times\mathscr{O}_X$.

Proposition 2.5. *Let X be a scheme with 2 invertible and $(\mathscr{E},q,\mathscr{L})$ be a regular line bundle-valued quadratic form of rank $n=2m$ and trivial discriminant. Then $[\mathscr{C}_0^+(\mathscr{E},q,\mathscr{L})]=[\mathscr{C}_0^-(\mathscr{E},q,\mathscr{L})]$ in ${}_2\mathrm{Br}(X)$.*

Proof. For m odd, the involution τ_0 is of unitary type with respect to the center (cf. [Auel1, Prop. 3.11]), hence induces an isomorphism

$$\mathscr{C}_0^+(\mathscr{E},q,\mathscr{L})\cong\mathscr{C}_0^-(\mathscr{E},q,\mathscr{L})^{\mathrm{op}}. \tag{10}$$

Hence it suffices to prove that $[\mathscr{C}_0^\pm(\mathscr{E},q,\mathscr{L})]$ are 2-torsion in $\mathrm{Br}(X)$. For this, we can appeal to the étale cohomological Tits algebra construction of [Auel1, Thm. 3.17].

For m even, the involution τ_0 is of the first kind and trivial on the center, restricting to involutions $\tau_0^\pm$ of the first kind on $\mathscr{C}_0^\pm(\mathscr{E},q,\mathscr{L})$ (in particular, they have 2-torsion Brauer classes). Thus, there exist refined classes $[\mathscr{C}_0^\pm(\mathscr{E},q,\mathscr{L}),\tau_0^\pm]$ in $H^2_{\text{ét}}(X,\boldsymbol{\mu}_2)$ lifting the Brauer classes $[\mathscr{C}_0^\pm(\mathscr{E},q,\mathscr{L})]$ in ${}_2\mathrm{Br}(X)$ and satisfying

$$[\mathscr{C}_0^+(\mathscr{E},q,\mathscr{L}),\tau_0^+]+[\mathscr{C}_0^-(\mathscr{E},q,\mathscr{L}),\tau_0^-]=c_1(\mathscr{L},\boldsymbol{\mu}_2),$$

see [Auel1, §2.8, §3.4], where $c_1(\mathscr{L},\boldsymbol{\mu}_2)\in H^2_{\text{ét}}(X,\boldsymbol{\mu}_2)$ is the 1st Chern class arising from the coboundary map of the Kummer squaring sequence. In particular, we have $[\mathscr{C}_0^+(\mathscr{E},q,\mathscr{L})]=[\mathscr{C}_0^-(\mathscr{E},q,\mathscr{L})]$ in ${}_2\mathrm{Br}(X)$ since 1st Chern classes are in the kernel of the natural map $H^2_{\text{ét}}(X,\boldsymbol{\mu}_2)\to H^2_{\text{ét}}(X,\mathbf{G}_{\mathrm{m}})$. □

The statement of Proposition 2.5 (and hence of Theorem 2.6, below) should remain true without the hypothesis that 2 is invertible on X. In the setting of Proposition 2.5, we will write $[\mathscr{C}_0^\pm(q)]=[\mathscr{C}_0^\pm(\mathscr{E},q,\mathscr{L})]$ for the Brauer class in question.

Theorem 2.6. *Let X be a scheme with 2 invertible and $(\mathscr{E},q,\mathscr{L})$ and $(\mathscr{E}',q',\mathscr{L})$ be regular line bundle-valued quadratic forms of even rank and trivial discriminant. Then $[\mathscr{C}_0^\pm(q\perp q')]=[\mathscr{C}_0^\pm(q)]+[\mathscr{C}_0^\pm(q')]$ in ${}_2\mathrm{Br}(X)$.*

Proof. Let e,f be complementary central splitting idempotents of $\mathscr{C}_0(q)$, inducing an $\mathscr{O}_X$-algebra decomposition

$$\mathscr{C}_0(q)=e\mathscr{C}_0(q)\times f\mathscr{C}_0(q)=\mathscr{C}_0^+(q)\times\mathscr{C}_0^-(q)$$

and a corresponding decomposition

$$\mathscr{C}_1(q)=\mathscr{C}_1(q)\cdot e\oplus\mathscr{C}_1(q)\cdot f=f\cdot\mathscr{C}_1(q)\oplus e\cdot\mathscr{C}_1(q)=\mathscr{C}_1^+(q)\oplus\mathscr{C}_1^-(q).$$

making $\mathscr{C}_1^\pm(q)$ into a $\mathscr{C}_0^\mp(q)$-$\mathscr{C}_0^\pm(q)$-bimodule via the $\mathscr{C}_0(q)$-bimodule structure on $\mathscr{C}_1(q)$. Local calculations, using Proposition 1.5b, shows that the map in Propositions 1.5c induces pairings

$$\mathscr{C}_1^\pm(q)\times\mathscr{C}_1^\mp(q)\to\mathscr{C}_0^\mp(q)\otimes\mathscr{L}. \tag{11}$$

Similarly, $\mathscr{C}_1^\pm(q)$ annihilates itself via the map in Propositions 1.5c, $\mathscr{C}_0^\pm(q)$ and $\mathscr{C}_0^\mp(q)$ annihilate each other via the multiplication in $\mathscr{C}_0(q)$, and the $\mathscr{C}_0^\pm(q)$-$\mathscr{C}_0^\mp(q)$-bimodule structure on $\mathscr{C}_1^\pm(q)$ induces via the $\mathscr{C}_0(q)$-bimodule structure on $\mathscr{C}_1(q)$, is zero.

Let e',f' be complementary central splitting idempotents of $\mathscr{C}_0(q')$, as above. Then $e\otimes e'+f\otimes f'$ and $e\otimes f'+f\otimes e'$ (via the isomorphism (8)) are complementary central splitting idempotents of $\mathscr{C}_0(q\perp q')$, inducing a decomposition

$$\mathscr{C}_0(q\perp q')=(e\otimes e'+f\otimes f')\mathscr{C}_0(q\perp q')\times(e\otimes f'+f\otimes e')\mathscr{C}_0(q\perp q')=\mathscr{C}_0^+(q\perp q')\times\mathscr{C}_0^-(q\perp q').$$

A direct local calculation, using the $\mathscr{C}_0^{\mp}(q)$-$\mathscr{C}_0^{\pm}(q)$-bimodule structure on $\mathscr{C}_1^{\pm}(q)$, the pairings (11), and the annihilation statements above, establishes the following block matrix algebra structures

$$\mathscr{C}_0^{+}(q \perp q') = \begin{pmatrix} \mathscr{C}_0^{+}(q) \otimes \mathscr{C}_0^{+}(q') & \mathscr{C}_1^{-}(q) \otimes \mathscr{C}_1^{-}(q') \otimes \mathscr{L}^{\vee} \\ \mathscr{C}_1^{+}(q) \otimes \mathscr{C}_1^{+}(q') \otimes \mathscr{L}^{\vee} & \mathscr{C}_0^{-}(q) \otimes \mathscr{C}_0^{-}(q') \end{pmatrix}$$
$$\mathscr{C}_0^{-}(q \perp q') = \begin{pmatrix} \mathscr{C}_0^{+}(q) \otimes \mathscr{C}_0^{-}(q') & \mathscr{C}_1^{-}(q) \otimes \mathscr{C}_1^{+}(q') \otimes \mathscr{L}^{\vee} \\ \mathscr{C}_1^{+}(q) \otimes \mathscr{C}_1^{-}(q') \otimes \mathscr{L}^{\vee} & \mathscr{C}_0^{-}(q) \otimes \mathscr{C}_0^{+}(q') \end{pmatrix}$$

via the isomorphism (8). The pairings (11) induce morphisms

$$\mathscr{C}_1^{\mp}(q) \cong \mathscr{H}om_{\mathscr{C}_0^{\pm}(q)}\big(\mathscr{C}_1^{\pm}(q), \mathscr{C}_0^{\pm}(q)\big) \otimes \mathscr{L}$$

of $\mathscr{C}_0^{\pm}(q)$-$\mathscr{C}_0^{\mp}(q)$-bimodules (these are right hom sheaves). Regularity implies that these are isomorphisms, with respect to which we have $\mathscr{O}_X$-algebra isomorphisms

$$\mathscr{C}_0^{+}(q \perp q') = \mathscr{E}nd_{\mathscr{C}_0^{+}(q) \otimes \mathscr{C}_0^{+}(q')}\big(\mathscr{C}_0^{+}(q) \otimes \mathscr{C}_0^{+}(q') \oplus \mathscr{C}_1^{+}(q) \otimes \mathscr{C}_1^{+}(q') \otimes \mathscr{L}^{\vee}\big)$$
$$\mathscr{C}_0^{-}(q \perp q') = \mathscr{E}nd_{\mathscr{C}_0^{+}(q) \otimes \mathscr{C}_0^{-}(q')}\big(\mathscr{C}_0^{+}(q) \otimes \mathscr{C}_0^{-}(q') \oplus \mathscr{C}_1^{+}(q) \otimes \mathscr{C}_1^{-}(q') \otimes \mathscr{L}^{\vee}\big).$$

In particular, $\mathscr{C}_0^{+}(q \perp q')$ is Brauer equivalent to $\mathscr{C}_0^{+}(q) \otimes \mathscr{C}_0^{+}(q')$ and $\mathscr{C}_0^{-}(q \perp q')$ is Brauer equivalent to $\mathscr{C}_0^{+}(q) \otimes \mathscr{C}_0^{-}(q')$. An application of Proposition 2.5 finishes the proof. □

When 2 is invertible on X, then by Theorems 1.7 and 2.6, for each line bundle $\mathscr{L}$ on X, the map $[\mathscr{E}, q, \mathscr{L}] \mapsto [\mathscr{C}_0^{+}(\mathscr{E}, q, \mathscr{L})]$ for any choice of central splitting idempotent, defines a homomorphism $e^2_{\mathscr{L}} : I^2(X, \mathscr{L}) \to {}_2\mathrm{Br}(X)$. By Proposition 1.2*e*, $\{e^2_{\mathscr{L}}\}$ is a system of projective similarity class invariants and there is a *total Clifford invariant* homomorphism

$$e^2 : I^2_{\mathrm{tot}}(X) \to {}_2\mathrm{Br}(X). \tag{12}$$

Remark 2.7. The invariant $e^2_{\mathscr{O}_X} : I^2(X) = I^2(X, \mathscr{O}_X) \to {}_2\mathrm{Br}(X)$ coincides with the classical Clifford invariant map. Indeed, if $(\mathscr{E}, q)$ is a regular $\mathscr{O}_X$-valued quadratic form of even rank and trivial discriminant then $\mathscr{C}_0^{+}(\mathscr{E}, q)$ is Brauer equivalent to the full Clifford algebra $\mathscr{C}(\mathscr{E}, q)$. See also [Auel1, Thm. 2.10*b*]. It was already proved in [KO] that the full Clifford algebra yields a homomorphism $W(X) \to {}_2\mathrm{Br}(X)$.

3. Surjectivity of the total Clifford invariant

The goal of this section is to prove Theorem A. Recall that an Azumaya algebra $\mathscr{A}$ over a scheme X has $\mathscr{O}_X$-rank d^2 for a positive integer d called the *degree*. The *period* of $\mathscr{A}$ is the order of the Brauer class $[\mathscr{A}] \in \mathrm{Br}(X)$. The *index* of $\mathscr{A}$ is the greatest common divisor of all degrees of Azumaya algebras $\mathscr{B}$ such that $\mathscr{A} \otimes \mathscr{E}nd\, \mathscr{P} \cong \mathscr{B} \otimes \mathscr{E}nd\, \mathscr{Q}$ for vector bundles $\mathscr{P}$ and $\mathscr{Q}$ on X. If X is integral with function field K, the *generic index* of $\mathscr{A}$ is the index of the central simple K-algebra $\mathscr{A}_K$. The generic index divides the index, with equality if X is regular of dimension ≤ 2. We will assume that 2 is invertible on X.

3.1. **Exceptional isomorphisms.** The exceptional isomorphisms of Dynkin diagrams $\mathsf{A}_1^2 = \mathsf{D}_2$ and $\mathsf{A}_3 = \mathsf{D}_3$ have beautiful reverberations in the theory of quadratic forms of rank 4 and 6, respectively. In these ranks, the *reduced norm* and *reduced pfaffian* constructions enable a quadratic form to be reconstructed from its even Clifford algebra (together with certain data). For quadratic forms over rings, this theory was initiated by Kneser, Knus, Ojanguren, Parimala, Paques, and Sridharan, see [Knes], [KOS], [KP], [Knu1], [KPS1], [KPS2]. Now, a standard reference on this work is Knus [Knu2, Ch. V]. Over fields, a wonderful reference is [KMRT, IV §15]. Bichsel [Bic] and Bichsel–Knus [BK] provide an extension of this theory to line bundle-valued forms over rings. The existing theory over rings immediately generalizes to base schemes when the corresponding algebraic groups are of inner type (i.e., the case of trivial discriminant). For an approach over general bases using Severi–Brauer schemes, see [PS2]. In the case of general discriminant, the details are worked out in [Auel1, §5].

We now outline the main results of this theory that we need. For even $n = 2m$, denote by $\mathrm{PQF}_n^+(X)$ the set of projective similarity classes of regular line bundle-valued quadratic forms of rank n and trivial discriminant on X. Denote by ${}_2\mathrm{Az}_d(X)$ the set of isomorphism classes of Azumaya $\mathscr{O}_X$-algebras of degree d and period 2.

For ease of exposition, and without loss of generality, we can assume that X is connected. The assignment, sending the projective similarity class of a quadratic form $(\mathscr{E}, q, \mathscr{L})$ of even rank $n = 2m$ and trivial discriminant to the unordered pair consisting of the $\mathscr{O}_X$-algebra isomorphism classes of the components $\mathscr{C}_0^+(\mathscr{E}, q, \mathscr{L})$ and $\mathscr{C}_0^-(\mathscr{E}, q, \mathscr{L})$ of the even Clifford algebra (for some central splitting idempotent), yields a well defined map

$$\mathrm{PQF}_n^+(X) \to {}_2\mathrm{Az}_{2^{m-1}}^{(2)}(X) \tag{13}$$

where $\{-\}^{(2)}$ denotes the set of unordered pairs of elements.

For any odd k, denote by ${}_2\mathrm{Az}'_{2^k}(X) \subset {}_2\mathrm{Az}^{(2)}_{2^k}(X)$ the subset of pairs of Brauer equivalent Azumaya algebras. For any even k, denote by ${}_2\mathrm{Az}'_{2^k}(X)$ the set of equivalence classes of Azumaya algebras of degree 2^k and period 2 under the relation $\mathscr{A} \sim \mathscr{B}$ if $\mathscr{A} \cong \mathscr{B}$ or $\mathscr{A} \cong \mathscr{B}^{\mathrm{op}}$. Then for even k, there is a canonical injective map ${}_2\mathrm{Az}'_{2^k}(X) \to {}_2\mathrm{Az}^{(2)}_{2^k}(X)$ given by $\mathscr{A} \mapsto (\mathscr{A}, \mathscr{A}^{\mathrm{op}})$.

For $n \equiv 0 \bmod 4$, recall that $\mathscr{C}_0^+(\mathscr{E}, q, \mathscr{L})$ is Brauer equivalent to $\mathscr{C}^-(\mathscr{E}, q, \mathscr{L})$ by Proposition 2.5. For $n \equiv 2 \bmod 4$, recall that $\mathscr{C}_0^+(\mathscr{E}, q, \mathscr{L}) \cong \mathscr{C}_0^-(\mathscr{E}, q, \mathscr{L})^{\mathrm{op}}$ by (10). Hence (13) factors through a map

$$\mathscr{C}_0^{\pm} : \mathrm{PQF}_n^+(X) \to {}_2\mathrm{Az}'_{2^{m-1}}(X). \tag{14}$$

The main result is that for $n = 4$ and $n = 6$, the map (14) is a bijection, with inverse map realized, respectively, by the reduced norm and pfaffian construction outlined in [KPS2], [Knu2, V.4–5], [PS2]. We now proceed to summarize these constructions.

Reduced norm form. In the $n = 4$ case, given a pair of Brauer equivalent Azumaya quaternion algebras $\mathscr{A}$ and $\mathscr{B}$, fibered Morita theory (cf. Lieblich [Lie1, §2.1.4] or Kashiwara–Schapira [KS, §19.5]) provides a $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathscr{P}$, which is invertible over $\mathscr{A}$ and $\mathscr{B}$ and is unique up to tensoring by a line bundle. Descending the reduced norm via étale splittings of $\mathscr{A}$ and $\mathscr{B}$, there exists a *reduced norm form* $\mathscr{N}(\mathscr{P}) = (\mathscr{P}, q_{\mathscr{P}}, \mathscr{N}_{\mathscr{P}})$, consisting of line bundle $\mathscr{N}_{\mathscr{P}}$ and a regular quadratic form $q_{\mathscr{P}} : \mathscr{P} \to \mathscr{N}_{\mathscr{P}}$ satisfying $q_{\mathscr{P}}(a \cdot p \cdot b) = \mathrm{Nrd}_{\mathscr{A}}(a)\, q_{\mathscr{P}}(p)\, \mathrm{Nrd}_{\mathscr{B}}(b)$ for sections a of $\mathscr{A}$, b of $\mathscr{B}$, and p of $\mathscr{P}$, where $\mathrm{Nrd}_{\mathscr{A}} : \mathscr{A} \to \mathscr{O}_X$ is the classical reduced norm. Tensoring $\mathscr{P}$ by a line bundle induces a projective similarity of reduced norm forms. Also, $\mathscr{P}$ is a $\mathscr{B}$-$\mathscr{A}$-bimodule by composing each action with the standard involution, giving rise to the same reduced norm form, hence we can freely exchange the role of $\mathscr{A}$ and $\mathscr{B}$.

Reduced pfaffian form. In the $n = 6$ case, given an Azumaya algebra $\mathscr{A}$ of degree 4 and period 2, there exists a vector bundle $\mathscr{P}$ of rank 16, unique up to tensoring by a line bundle, and an $\mathscr{O}_X$-algebra isomorphism $\varphi : \mathscr{A} \otimes \mathscr{A} \cong \mathscr{E}nd(\mathscr{P})$. The reduced trace, considered as an element of $\mathscr{E}nd\,\mathscr{A} \cong \mathscr{A}^{\mathrm{op}} \otimes \mathscr{A}$, is mapped via φ to an involutory $\mathscr{O}_X$-module endomorphism $\psi : \mathscr{P} \to \mathscr{P}$. The subsheaf $A_\psi(\mathscr{P}) = \mathrm{im}(\mathrm{id}_{\mathscr{P}} - \psi)$ of alternating elements with respect to ψ is a vector bundle of rank 6, as can be checked étale locally. Descending the pfaffian map via étale splitting of $\mathscr{A}$ and $\mathscr{P}$, there exists a *reduced pfaffian form* $\mathscr{P}\!f(\mathscr{P}) = (A_\psi(\mathscr{P}), \mathrm{pf}_{\mathscr{P}}, \mathscr{P}\!f_{\mathscr{P}})$, consisting of a line bundle $\mathscr{P}\!f_{\mathscr{P}}$ and a regular quadratic form $\mathrm{pf}_{\mathscr{P}} : A_\psi(\mathscr{P}) \to \mathscr{P}\!f_{\mathscr{P}}$. Tensoring $\mathscr{P}$ by a line bundle tensors $A_\psi(\mathscr{P})$ by the square of the line bundle, inducing a projective similarity of reduced pfaffian forms. Exchanging $\mathscr{A}$ with $\mathscr{A}^{\mathrm{op}}$ replaces $\mathscr{P}$ by $\mathscr{P}^\vee$ and ψ by $\psi^\vee$, giving rise to isomorphisms $A_{\psi^\vee}(\mathscr{P}^\vee) \cong A_\psi(\mathscr{P})^\vee$ and $\mathscr{P}\!f_{\mathscr{P}^\vee} \cong (\mathscr{P}\!f_{\mathscr{P}})^\vee$ (cf. [Knu2, III Lemma 9.3.5]) and a projective similarity of reduced pfaffian forms $\mathscr{P}\!f(\mathscr{P})$ and $\mathscr{P}\!f(\mathscr{P}^\vee)$ (cf. [Knu2, III Prop. 9.4.2]).

Theorem 3.1. *Let X be a scheme with 2 invertible.*

a) There are inverse bijections

$$\mathrm{PQF}_4^+(X) \underset{\mathscr{N}}{\overset{\mathscr{C}_0^\pm}{\rightleftarrows}} {}_2\mathrm{Az}_2'(X)$$

where $\mathscr{N}$ is the reduced norm form construction.

b) There are inverse bijections

$$\mathrm{PQF}_6^\pm(X) \underset{\mathscr{P}\!f}{\overset{\mathscr{C}_0^+}{\rightleftarrows}} {}_2\mathrm{Az}_4'(X)$$

where $\mathscr{P}\!f$ is the reduced pfaffian form construction.

Proof. Given Brauer equivalent Azumaya algebras $\mathscr{A}$ and $\mathscr{B}$, there exists an invertible $\mathscr{A}$-$\mathscr{B}$-bimodule $\mathscr{P}$ such that $\mathscr{B} \cong \mathscr{E}nd_{\mathscr{A}}(\mathscr{P})$. Hence for m even (e.g., $m = 2$), ${}_2\mathrm{Az}_{2^{m-1}}'(X)$ is in bijection with the set of isomorphism classes of pairs $(\mathscr{A}, \mathscr{P})$, consisting of an Azumaya algebra $\mathscr{A}$ of degree n and an invertible right $\mathscr{A}$-module $\mathscr{P}$. A direct proof of *a* can be deduced from [PS2, Prop. 4.1] (itself a generalization of [BK, Prop. 4.5]), which states that if $\mathscr{A}$ is an Azumaya quaternion algebra and $\mathscr{P}$ is an invertible right $\mathscr{A}$-module, then $\mathscr{C}_0(\mathscr{N}(\mathscr{P})) \cong \mathscr{A} \times \mathscr{E}nd_{\mathscr{A}}(\mathscr{P})$. By [PS2, Prop. 4.3], the map $\mathscr{N}$ is surjective. Hence $\mathscr{N}$ and $\mathscr{C}_0^\pm$ are inverse bijections. This is a generalization of [KPS2, Thm. 10.7] to the line bundle-valued (trivial discriminant) setting.

A direct proof of *b* can be given along similar lines. By [BK, Prop. 4.8] (which immediately generalizes to general base schemes), if $\mathscr{A}$ is an Azumaya $\mathscr{O}_X$-algebra of degree 4, $\mathscr{P}$ is a locally free $\mathscr{O}_X$-module of rank 16, and $\varphi : \mathscr{A} \otimes \mathscr{A} \to \mathscr{E}nd(\mathscr{P})$ is an $\mathscr{O}_X$-algebra isomorphism (corresponding to the element $[\mathscr{A}] \in {}_2\mathrm{Az}_4'(X)$), then $\mathscr{C}_0(\mathscr{P}\!f(\mathscr{P})) \cong \mathscr{A}^{\mathrm{op}} \times \mathscr{E}nd_{\mathscr{A}^{\mathrm{op}}}(\mathscr{P})$. By [PS2, Prop. 6.1], the map $\mathscr{P}\!f$ is surjective. Hence $\mathscr{P}\!f$ and $\mathscr{C}_0^\pm$ are inverse maps. This is a generalization of [KPS2, Thm. 9.4] to the line bundle-valued (trivial discriminant) setting. □

As a result, we can realize any Azumaya algebra of degree dividing 4 on X as the even Clifford invariant of a line bundle-valued quadratic form. In particular, if ${}_2\mathrm{Br}(X)$ is generated by such Azumaya algebras, then the total Clifford invariant is surjective.

Corollary 3.2. *Let X be a scheme with 2 invertible. If ${}_2\mathrm{Br}(X)$ is generated by Azumaya algebras of degree ≤ 4, then the total Clifford invariant*

$$e^2 : I^2_{\mathrm{tot}}(X) \to {}_2\mathrm{Br}(X)$$

is surjective.

Note that if X is the spectrum of a field, then ${}_2\mathrm{Br}(X)$ is always generated by quaternion algebras by Merkurjev's theorem, hence the hypotheses of Corollary 3.2 are quite global in nature.

In the same spirit, we can give a stronger condition sufficient for the surjectivity of the classical Clifford invariant $e^2_{\mathscr{O}_X} : I^2(X) \to {}_2\mathrm{Br}(X)$. First we recall some results from [KPS2]. Let $[\mathscr{A}] \in {}_2\mathrm{Az}_4(X)$ have reduced pfaffian form $(A_\psi(\mathscr{P}), \mathrm{pf}_{\mathscr{P}}, \mathscr{P}\!f_{\mathscr{P}})$, choosing a vector bundle $\mathscr{P}$ of rank 16 such that $\mathscr{A} \otimes \mathscr{A} \cong \mathscr{E}nd\,\mathscr{P}$. The class $d_0(\mathscr{A}) = [\mathscr{P}\!f_{\mathscr{P}}] \in \mathrm{Pic}(X)/2$ is a well defined invariant of $\mathscr{A}$, see [KPS2, §9, p. 213]. When $d_0(\mathscr{A})$ is trivial we say that $\mathscr{A}$ has *trivial pfaffian invariant*.

Proposition 3.3 ([KPS2, Prop. 3.2])**.** *Let X be a scheme with 2 invertible and $\mathscr{A} \in {}_2\mathrm{Az}_4(X)$. If $\mathscr{A}$ has an involution of the first kind then $d_0(\mathscr{A})$ is 2-torsion. Moreover, if $\mathscr{A}$ has a symplectic involution then $d_0(\mathscr{A})$ is trivial.*

We recall that any Azumaya quaternion algebra has a standard symplectic involution, hence has trivial pfaffian invariant.

Corollary 3.4. *Let X be a scheme with 2 invertible. If ${}_2\mathrm{Br}(X)$ is generated by Azumaya algebras of degree dividing 4 with trivial pfaffian invariant, then the classical Clifford invariant*

$$e^2_{\mathscr{O}_X} : I^2(X) \to {}_2\mathrm{Br}(X)$$

is surjective. In particular this is the case if ${}_2\mathrm{Br}(X)$ is generated by Azumaya quaternion algebras.

Proof. We first remark that any $\mathscr{A} \in {}_2\mathrm{Az}_4(X)$ of index 2 is Brauer equivalent to $\mathscr{A}' \in {}_2\mathrm{Az}_4(X)$ with trivial pfaffian invariant. Indeed, if $\mathscr{A}$ has index 2, then $\mathscr{A} \cong \mathscr{E}nd_{\mathscr{B}}(\mathscr{P})$ for an Azumaya quaternion algebra $\mathscr{B}$ and a locally free $\mathscr{B}$-module $\mathscr{P}$ of rank 2. We can extend the standard symplectic involution on $\mathscr{B}$ to $\mathscr{A}' = \mathscr{M}_2(\mathscr{B})$, which then has trivial pfaffian invariant by Proposition 3.3. But $\mathscr{A}$ is Brauer equivalent to $\mathscr{A}'$.

Now, note that the reduced norm form $q_{\mathscr{B}} : \mathscr{B} \to \mathscr{O}_X$ is a regular $\mathscr{O}_X$-valued quadratic form in $I^2(X)$ with $e^2_{\mathscr{O}_X}(\mathscr{N}(\mathscr{B})) = [\mathscr{B}]$, by Theorem 3.1*a*. This already proves the final claim. In general, if $\mathscr{A} \in {}_2\mathrm{Az}_4(X)$ has trivial pfaffian invariant, then there exists an $\mathscr{O}_X$-valued quadratic form $(\mathscr{E}, q)$ in the projective similarity class of $\mathscr{P}f(\mathscr{A})$. By Theorem 3.1*b*, we have that $e^2_{\mathscr{O}_X}(\mathscr{E}, q) = [\mathscr{A}]$. The first claim follows. $\square$

3.2. **Brauer dimension results.** Now we investigate sufficient conditions under which ${}_2\mathrm{Br}(X)$ is generated by Azumaya algebras of degree dividing 4. Let X be an integral scheme with function field K. An Azumaya $\mathscr{O}_X$-algebra $\mathscr{A}$ is called an *Azumaya division algebra* if the generic fiber $\mathscr{A}_K$ is a central division K-algebra.

We introduce two conditions on an integral scheme X with function field K:

A Every central division K-algebra of period 2 and degree dividing 4, which is Brauer equivalent to the generic fiber of an Azumaya $\mathscr{O}_X$-algebra, is isomorphic to the generic fiber of an Azumaya division $\mathscr{O}_X$-algebra, i.e., restriction to the generic point ${}_2\mathrm{Az}_d(X) \to {}_2\mathrm{Az}_d(K)$ is surjective for d dividing 4.

B Every $\mathscr{A} \in {}_2\mathrm{Br}(X)$ satisfies $\mathrm{index}(\mathscr{A}_K) \,|\, \mathrm{period}(\mathscr{A}_K)^2$, i.e., $\mathrm{index}(\mathscr{A}_K) \,|\, 4$.

Condition A is a kind of "purity for division algebras" of period 2 and degree dividing 4, or "purity for $\mathbf{GL}_4/\mu_2$-torsors" in the setting of Colliot-Thélène–Sansuc [CTS2]. Condition B might be restated loosely as "X has Brauer dimension 2" for classes of period 2. See [ABGV, §4] for the precise notion of Brauer dimension.

We now prove that under conditions A and B, we get an "unramified symbol length" bound on the Brauer group, which is stronger than the generation hypothesis needed for Corollary 3.2.

Theorem 3.5. *Let X be a regular integral scheme with 2 invertible. If X satisfies conditions A and B, then ${}_2\mathrm{Br}(X)$ is represented by Azumaya algebras of degree dividing 4. In particular, the total Clifford invariant is surjective.*

Proof. Since X is regular, the canonical map $\mathrm{Br}(X) \to \mathrm{Br}(K)$ is injective, see [AG] or [Gro, Cor. 1.8]. By condition B, for any $\mathscr{A} \in {}_2\mathrm{Br}(X)$, we have that $\mathscr{A}_K \in {}_2\mathrm{Br}(K)$ is Brauer equivalent to a central division K-algebra D of degree dividing 4. By condition A, there exists an Azumaya $\mathscr{O}_X$-algebra $\mathscr{B}$ whose generic fiber is D, in particular, $\mathscr{B}$ has degree dividing 4. Since $[\mathscr{B}_K] = [D] = [\mathscr{A}_K] \in {}_2\mathrm{Br}(K)$, by the injectivity of $\mathrm{Br}(X) \to \mathrm{Br}(K)$, we have that $[\mathscr{B}] = [\mathscr{A}] \in {}_2\mathrm{Br}(X)$. The final claim is thus a direct consequence of Corollary 3.2. $\square$

We now collect together some necessary conditions under which conditions A and B hold. Condition A (and more generally, purity for division algebras of any degree) is satisfied quite generally for schemes of dimension ≤ 2.

Theorem 3.6. *Any regular integral scheme X of dimension ≤ 2 satisfies condition A.*

Proof. Apply Colliot-Thélène–Sansuc [CTS2, Cor. 6.14] to the reductive group scheme $\mathbf{GL}_4/\mu_2$ over X. An alternate proof can be found in [APS, Thm. 4.3]. □

Note that for schemes of higher dimension, Condition A can fail, see [AW3]

As for condition B, it holds in the following cases where the Brauer dimension of K is known to be 1:

- smooth curves over finite fields (by class field theory),
- smooth surfaces over algebraically closed fields (by Artin [Art] or de Jong [dJ]),

and where the Brauer dimension of K is known to be 2:

- smooth curves over local fields (by Saltman [Sal2]),
- smooth surfaces over (pseudo-)finite fields (by Lieblich [Lie2]).

We can now proceed to prove Theorem A.

Corollary 3.7. *Let X be regular integral scheme with 2 invertible.*

a) If X is a smooth curve over a finite field or surface over an algebraically closed field, then the classical Clifford invariant $e^2 : I^2(X) \to {}_2\mathrm{Br}(X)$ is surjective.
b) If X is a smooth curve over a local field or a surface over a (pseudo-)finite field, then the total Clifford invariant $e^2 : I^2_{\mathrm{tot}}(X) \to {}_2\mathrm{Br}(X)$ is surjective.

Proof. This is a direct consequence of Corollaries 3.2 and 3.4, Theorem 3.5, Theorem 3.6, and the Brauer dimension results stated above. Note that *a* was already known for curves over finite fields by [PS1, Lemma 4.1] and for surfaces over algebraically closed fields by [FC]. □

We remark that recent results of Lieblich–Parimala–Suresh [LPS] imply that, assuming a conjecture of Colliot-Thélène on the Brauer–Manin obstruction to the existence of 0-cycles of degree 1 on smooth projective varieties over global fields, condition B also holds for regular arithmetic surfaces, i.e., regular schemes proper and flat over the spectrum of the ring of integers of a number field whose generic fiber is a geometrically connected curve. Thus Theorem A holds conditionally for regular arithmetic surfaces. Also, recent results of Harbater–Hartmann–Krashen [HHK14] prove condition B for a wide class of local curves over complete discrete valuation rings with finite or algebraically closed residue fields.

3.3. **A total unramified Milnor question.** We are lead to the following natural question, inspired by our main result.

Question 3.8. *Let X be a regular integral scheme with 2 invertible. Assume that the function field K of X satisfies $cd_2(K) \leq 3$. Is the homomorphism*

$$e^2 : I^2_{\mathrm{tot}}(X) \to {}_2\mathrm{Br}(X)$$

surjective?

A positive answer to Question 3.8 brings a scheme closer to having a positive answer to an analogue of the unramified Milnor question for the fundamental filtration $I^2_{\mathrm{tot}}(X) \subset I^1_{\mathrm{tot}}(X) \subset W_{\mathrm{tot}}(X)$ of the total Witt group; see [Auel3, Question 3.1] for a survey of results on the unramified Milnor question. All schemes appearing in Corollary 3.7 have a positive answer to Question 3.8.

There are recent examples of Antieau–Williams [AW2, §7], [AW1, Example 3.13] of smooth affine schemes over $\mathbb{C}$ of dimension 5 with nonsurjective total Clifford invariant (these examples actually have nonsurjective classical Clifford invariant and trivial Picard group).

References

[AW1] B. Antieau and B. Williams, *Topology and purity for torsors*, Documenta Math. **20** (2015), 333–355.

[AW2] ———, *The topological period-index problem over 6-complexes*, J. Top. **7** (2014), 617–640.

[AW3] ———, *Unramified division algebras do not always contain Azumaya maximal orders*, Invent. Math. **197** (2014) 47–56.

[Ara] J. K. Arason, *A proof of Merkurjev's theorem*, Quadratic and Hermitian forms (Hamilton, Ont., 1983), CMS Conf. Proc., vol. 4, Amer. Math. Soc., Providence, RI, 1984, pp. 121–130.

[Art] M. Artin, *Brauer-Severi varieties*, Brauer groups in ring theory and algebraic geometry (Wilrijk, 1981), Lecture Notes in Math., vol. 917, Springer, Berlin, 1982, pp. 194–210.

[AG] M. Auslander and O. Goldman, *The Brauer group of a commutative ring*, Trans. Amer. Math. Soc. **97** (1960), 367–409.

[Auel1] A. Auel, *Clifford invariants of line bundle-valued quadratic forms*, MPIM preprint series 2011-33, 2011.

[Auel2] ———, *Vector bundles of rank 4 and* $\mathsf{A}_3 = \mathsf{D}_3$, Int. Math. Res. Notices **15** (2013), 3450–3476.

[Auel3] ———, *Remarks on the Milnor conjecture over schemes*, Galois-Teichmüller Theory and Arithmetic Geometry (Kyoto, 2010), H. Nakamura, F. Pop, L. Schneps, A. Tamagawa eds., Adv. Stud. Pure Math., vol. 63, 2012, pp. 1-30.

[ABGV] A. Auel, E. Brussel, S. Garibaldi, and U. Vishne, *Open problems in central simple algebras*, Transform. Groups **16** (2011), no. 1, 219–264.

[APS] A. Auel, R. Parimala, and V. Suresh, *Quadric surface bundles over surfaces*, Documenta Math., to appear.

[Bal] V. Balaji, *Line-bundle valued ternary quadratic forms over schemes*, J. Pure Appl. Algebra **208** (2007), 237–259.

[BC] P. Balmer and B. Calmès, *Bases of total Witt groups and lax-similitude*, J. Algebra Appl. **11** (2012), no. 03.

[Bic] W. Bichsel, *Quadratische raeume mit werten in invertierbaren moduln*, Ph.D. thesis, ETH Zürich, 1985.

[BK] W. Bichsel and M.-A. Knus, *Quadratic forms with values in line bundles*, Contemp. Math. **155** (1994), 293–306.

[CTS2] ———, *Fibrés quadratiques et composantes connexes réelles*, Math. Ann. **244** (1979), no. 2, 105–134.

[CvO] S. Caenepeel and F. van Oystaeyen, *Quadratic forms with values in invertible modules*, K-Theory **7** (1993), 23–40.

[DG] M. Demazure and P. Gabriel, *Groupes algébriques. Tome I: Géométrie algébrique, généralités, groupes commutatifs. Avec un appendice,* Corps de classes local*, par Michiel Hazewinkel*, Masson & Cie, Éditeur, Paris; North-Holland Publishing Company, Amsterdam, 1970.

[SGA3] M. Demazure and A. Grothendieck, *Schémas en groupes. I, II, III*, Séminaire de Géométrie Algébrique du Bois Marie 1962/64 (SGA 3), vol. 151, 152, 153, Springer-Verlag, New York, 1970.

[dJ] A. J. de Jong, *The period-index problem for the Brauer group of an algebraic surface*, Duke Math. J. **123** (2004), no. 1, 71–94.

[EKM] R. Elman, N. Karpenko, and A. Merkurjev, *The algebraic and geometric theory of quadratic forms*, American Mathematical Society Colloquium Publications, vol. 56, American Mathematical Society, Providence, RI, 2008.

[EVMS] P. Elbaz-Vincent and S. Müller-Stach, *Milnor K-theory of rings, higher Chow groups and applications*, Invent. Math. **148** (2002), no. 1, 177–206.

[FC] F. Fernández-Carmena, *On the injectivity of the map of the Witt group of a scheme into the Witt group of its function field*, Math. Ann. **277** (1987), no. 3, 453–468.

[GHKS] W.-D. Geyer, G. Harder, M. Knebusch, and W. Scharlau, *Ein Residuensatz für symmetrische Bilinearformen*, Invent. Math. **11** (1970), 319–328.

[Gil] S. Gille, *A graded Gersten-Witt complex for schemes with a dualizing complex and the Chow group*, J. Pure Appl. Algebra **208** (2007), no. 2, 391–419.

[Gro] A. Grothendieck, *Le groupe de Brauer. II. Théorie cohomologique*, Dix Exposés sur la Cohomologie des Schémas, North-Holland, Amsterdam, 1968, pp. 67–87.

[Gui] D. Guin, *Homologie du groupe linéaire et K-théorie de Milnor des anneaux*, J. Algebra **123** (1989), no. 1, 27–59.

[HHK14] D. Harbater, J. Hartmann, D. Krashen, *Refinements to patching and applications to field invariants*, Int. Math. Res. Notices (2015), imrn/rnu278.

[Hoo] R. Hoobler, *The Merkuriev-Suslin theorem for any semi-local ring*, J. Pure Appl. Algebra **207** (2006), no. 3, 537–552.

[Kan] T. Kanzaki, *On bilinear module and Witt ring over a commutative ring*, Osaka J. Math. **8** (1971), 485–496.

[Kap] M. Kapranov, *On the derived categories of coherent sheaves on some homogeneous spaces*, Invent. Math. **92** (1988), 479–508.

[KS] M. Kashiwara and P. Schapira, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 332, Springer-Verlag, Berlin, 2006.

[Ker] M. Kerz, *The Gersten conjecture for Milnor K-theory*, Invent. Math. **175** (2009), no. 1, 1–33.

[KMS] M. Kerz and S. Müller-Stach, *The Milnor-Chow homomorphism revisited*, K-Theory **38** (2007), no. 1, 49–58.

[Kneb2] ———, *Symmetric bilinear forms over algebraic varieties*, Conference on Quadratic Forms–1976 (Kingston, Ont.) (G. Orzech, ed.), Queen's Papers in Pure and Appl. Math., no. 46, Queen's Univ., 1977, pp. 103–283.

[Knes] M. Kneser, *Composition of binary quadratic forms*, J. Number Theory **15** (1982), no. 3, 406–413.

[Knu1] M.-A. Knus, *Pfaffians and quadratic forms*, Adv. in Math. **71** (1988), 1–20.

[Knu2] ———, *Quadratic and hermitian forms over rings*, Springer-Verlag, Berlin, 1991.

[KMRT] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, Colloquium Publications, vol. 44, AMS, 1998.

[KO] M.-A. Knus and M. Ojanguren, *The Clifford algebra of a metabolic space*, Arch. Math. (Basel) **56** (1991), no. 5, 440–445.

[KOS] M.-A. Knus, M. Ojanguren, and R. Sridharan, *Quadratic forms and Azumaya algebras*, J. Reine Angew. Math. **303/304** (1978), 231–248.

[KP] M.-A. Knus and A. Paques, *Quadratic spaces with trivial Arf invariant*, J. Algebra **93** (1985), no. 2, 267–291.

[KPS1] M.-A. Knus, R. Parimala, and R. Sridharan, *On rank 4 quadratic spaces with given Arf and Witt invariants*, Math. Ann. **274** (1986), no. 2, 181–198.

[KPS2] ———, *A classification of rank 6 quadratic spaces via pfaffians*, J. reine angew. Math. **398** (1989), 187–218.

[Kuz] A. Kuznetsov, *Derived categories of quadric fibrations and intersections of quadrics*, Adv. Math. **218** (2008), no. 5, 1340–1369.

[Lie1] M. Lieblich, *Moduli of twisted sheaves and generalized Azumaya algebras*, Ph.D. thesis, Massachusetts Institute of Technology, Cambridge, MA, June 2004.

[Lie2] ———, *The period-index problem for fields of transcendence degree 2*, Ann. of Math., to appear.

[LPS] M. Lieblich, R. Parimala, and V. Suresh, *Colliot-Thélène's conjecture and finiteness of u-invariants*, Math. Ann. **360** (2014), no. 1-2, 1–22.

[Mah] M. Mahmoudi, *Orthogonal symmetries and Clifford algebras*, Proc. Indian Acad. Sci. Math. Sci. **120** (2010), no. 5, 535–561.

[Mer1] A. Merkurjev, *On the norm residue symbol of degree* 2, Dokl. Akad. Nauk SSSR **261** (1981), no. 3, 542–547.

[Mer2] ———, *On the norm residue homomorphism of degree two*, Proceedings of the St. Petersburg Mathematical Society. Vol. XII (Providence, RI), Amer. Math. Soc. Transl. Ser. 2, vol. 219, Amer. Math. Soc., 2006, pp. 103–124.

[Mor] F. Morel, *Milnor's conjecture on quadratic forms and mod 2 motivic complexes*, Rend. Sem. Mat. Univ. Padova **114** (2005), 63–101 (2006).

[Mum] D. Mumford, *Theta characteristics of an algebraic curve*, Ann. Sci. École Norm. Sup. **4** (1971), no. 4, 181–192.

[PSch] R. Parimala and W. Scharlau, *On the canonical class of a curve and the extension property for quadratic forms*, Recent advances in real algebraic geometry and quadratic forms (Berkeley, CA, 1990/1991; San Francisco, CA, 1991), Contemp. Math., vol. 155, Amer. Math. Soc., Providence, RI, 1994, pp. 339–350.

[PS1] R. Parimala and R. Sridharan, *Graded Witt ring and unramified cohomology*, K-Theory **6** (1992), no. 1, 29–44.

[PS1] ———, *Nonsurjectivity of the Clifford invariant map*, Proc. Indian Acad. Sci. Math. Sci. **104** (1994), no. 1, 49–56, K. G. Ramanathan memorial issue.

[PS2] ———, *Reduced norms and pfaffians via Brauer-Severi schemes*, Contemp. Math. **155** (1994), 351–363.

[Sal1] D. Saltman, *Azumaya algebras with involution*, J. Algebra **52** (1978), no. 2, 526–539.

[Sal2] ———, *Division algebras over p-adic curves*, J. Ramanujan Math. Soc. **12** (1997), 25–47, errata, J. Ramanujan Math. Soc. **12** (1997), no. 1, 25–47.

[Voi] J. Voight, *Characterizing quaternion rings over an arbitrary base*, J. Reine Angew. Math. **657** (2011), 113–134.

[Wad] A. Wadsworth, *Merkurjev's elementary proof of Merkurjev's theorem*, Applications of algebraic K-theory to algebraic geometry and number theory, Part I, II (Boulder, Colo., 1983), Contemp. Math., vol. 55, Amer. Math. Soc., Providence, RI, 1986, pp. 741–776.

Department of Mathematics, Yale University, 10 Hillhouse Ave., New Haven, CT 06511
E-mail address: asher.auel@yale.edu